# FUSION OF p-ELEMENTS IN SYMMETRIC GROUPS

BY

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TO MY FATHER

TENG-CHUAN HSIEH

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## SYMBOLS

The following expressions concerning a finite group G will be used frequently:

| | |
|---|---|
| $E$ | identity element of G |
| $\lvert A \rvert$ | the order of a set A |
| $A \subset G$ | A is a subset of G |
| $A \leq G$ | A is a subgroup of G |
| $A \trianglelefteq G$ | A is a normal subgroup of G |
| $Z(G)$ | center of G |

For $H \leq G$, we define:

| | |
|---|---|
| $N(H)$ | normalizer of H in G |
| $C(H)$ | centralizer of H in G |
| $[G:H]$ | index of H in G |

For $H, K \leq G$, we define:

| | |
|---|---|
| $N_H(K)$ | $N(K) \cap H$ |
| $C_H(K)$ | $C(K) \cap H$ |

For $N \trianglelefteq G$, we define:

| | |
|---|---|
| $G/N$ | the factor group of G over N |

We also define; for $A, B \subset G$, $H, K, L \leq G$, $g, h, k \in G$:

| | |
|---|---|
| $A-B$ | set of all elements in A not in B |
| $\langle A \rangle$ | subgroup of G generated by A |
| $h^g$ | $g^{-1}hg$ |
| $A^g$ | $g^{-1}Ag$ |
| $[g,h]$ | $g^{-1}h^{-1}gh$ |
| $[H,K]$ | $\langle [h,k] \; h \in H, \; k \in K \rangle$ |
| $G'$ | $[G,G]$ |

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In a finite group G whose order is divisible by a prime p, the normalizers of p-subgroups are called local subgroups of G. The significance of local subgroups in the discussion concerning the fusion of p-elements, hence the relations between local and global properties of G, has been recognized in recent years.

In this paper the author defines a local fusion family of a Sylow p-subgroup S of G to be a family containing normalizers of some p-subgroups of S such that if any two elements of S are conjugate in G then the conjugating element can be found as a product of elements from local subgroups of the family.

It is proved, as the main theorem of this paper, that in the Sylow p-subgroups of a symmetric group of p-power degree, the normalizers of p-subgroups of only two different structures are needed to form a local fusion family. In fact, they are the normalizers of a large elementary abelian group and some small elementary abelian groups of order $p^2$.

Besides the main theorem which is proved in two parts, the structures of the normalizers of the two p-subgroups mentioned above are described in full detail.

CHAPTER I

# INTRODUCTION

## 1. Simple Groups and Finite Groups

In the study of finite groups much of the work centers on the study of finite simple groups, that is partially because simple groups to finite groups are just like prime numbers to natural numbers. Knowing all about simple groups would tell much about all finite groups. The Jordan-Hölder theorem for finite groups says: For every group G there is a sequence of subgroups $G=G_0 \trianglerighteq G_1 \trianglerighteq \dots G_{r-1} \trianglerighteq G_r = E$, such that each quotient group $G_i/G_{i+1}$ is a simple group and the collection of associated simple quotient groups is unique up to reordering. This is somewhat analogous to the Fundamental Theorem of Arithmetic: For every positive integer $n \neq 1$ there is a sequence $n=n_0 \geq n_1 \geq n_2 \geq \dots \geq n_{r-1} \geq n_r = 1$ such that each $n_i/n_{i+1}$ is a prime and the collection of primes which so occur and their multiplicities are uniquely determined up to possible renumbering of indices $i$.

Suppose all finite simple groups were known. How could one use this and the Jordan-Hölder Theorem to determine all finite groups? First one would get a list of simple groups $S_1, S_2, \dots, S_r$, then would try to build up a tree diagram listing the possible groups $G_i$ such that $G_i/G_{i+1} \approx S_{i+1}$, ending with all possible $G=G_0$ whose composition factors are

the simple groups $S_1, S_2, \ldots, S_r$. To proceed doing this, in each step one would have to know how to determine all possible $G_i$ by knowing $G_{i+1}$ and $G_i/G_{i+1} \approx S_{i+1}$. This question is the so called Extension Problem which can be stated in a more general way: Given two groups N and H, determine all possible groups G such that $N \trianglelefteq G$ and $G/N \simeq H$. Such groups G are called extensions of N by the factor H. All possible such extensions G can be constructed in some systematic way. Schreider [19] developed a technique using automorphisms of N and a factor set to construct an extension G by obtaining the multiplication table for G. This method can also be translated into the language of cohomological algebra in the case that N is abelian. (See Rotman [18] §5 and §10 or Huppert [13] Ch I, §16-17.) Thus Schreider's method and the Jordan-Hölder Theorem allow us to construct all possible finite groups, although not very practically, from all finite simple groups. This is why simple groups are so important in finite group theory.

## 2. Local and Global Properties of Finite Groups

For every subgroup H of a finite group G, the structure of H, such as the list of elements and subgroups and the equations of the multiplication table of H, or the embedding of H in G are part of the local structure of G. While the normal subgroups, quotient groups and conjugacy classes are relevant to the global properties of G. The relations between the local and global properties of a finite group are very important. In the study of these

relations the representation theory and transfer are very useful. The application of these techniques is often dependent on results concerning the fusion of elements. The study of fusion of p-elements, where p is a prime, is also very close related to the question of whether a given group G possesses a nontrivial p-factor group, which is in turn related to the question of simplicity of finite groups.

Higman (1953), using the transfer, proved the focal subgroup theorem; see [12]:

(2.1) If P is a Sylow p-subgroup of a finite group G, then $P \cap G'$ is generated by all elements of the form $a^{-1}b$ where a and b are elements of P conjugate in G.

This theorem shows that the fusion of elements of P determines the focal subgroup $P \cap G'$, hence $P/P \cap G'$, which is isomorphic to the largest abelian p-factor group of G. Thus the fusion of elements of P plays an important role in connections of local and global properties of G.

The question of how two elements of P are fused in G has a very satisfactory answer stated in Alperin's fusion theorem [1]:

(2.2) If P is a Sylow p-subgroup of G and a,b are elements of P conjugate in G then there exist elements $a=a_1, a_2, \ldots, a_m=b$ and subgroups $H_1, H_2, \ldots, H_{m-1}$ of P such that $a_i, a_{i+1}$ lie in $H_i$ and are conjugate in the normalizer $N(H_i)$ of $H_i$, $i=1,2,\ldots,m-1$.

The well-known theorems of Burnside, Frobenius and Grün giving conditions for the existence of nontrival p-factor groups can be proved much easier by the focal subgroup theorem and Alperin's fusion theorem. For more

detail about the proofs of these theorems, see Gorenstein [9] Ch 7.

If we call the normalizer of a p-subgroup a <u>local subgroup</u>, then Alperin theorem can be interpreted as: If two elements of a group are conjugate, then the conjugating element factors into a finite number of elements, each lying in a local subgroup. Each such element acts as a <u>local conjugation</u>.

The significance of local subgroups in the fusion of p-elements can be seen further in Alperin's recent article [3]. Other questions such as the number of <u>local conjugation steps</u> and the family of p-subgroups whose normalizers give all local fusion were discussed extensively by some people. As to the first question, Alperin's conjecture <u>that the number of local conjugation steps is bounded by a function which depends only on the nilpotency class of P</u> is proved by Dolan in [5]. Alperin in [2] (1974) also proved that in his theorem, the elements $a_1, a_2, \ldots, a_m$ can be chosen so that the orders of their <u>centralizers</u> in P first increase monotonically and then decrease monotonically; this is called "up and down fusion". As to the second question, Alperin in [1] also defined a <u>conjugation family</u> and weak <u>conjugation family</u> as follows:

(2.3) <u>Let P be a Sylow p-subgroup of G, a conjugation family F (resp. weak conjugation family) for P is a collection of pairs (H,T), where $H \leq P$ and $T \leq N(H)$, satisfying the following condition: For any two subsets A, B of P with $B=A^g$ for some g in G, there exist $(H_1,T_1),(H_2,T_2),\ldots,$</u>

$(H_n, T_n)$ in F and elements $x_1, x_2, \ldots, x_n, y$ in G such that

(a) $g = x_1 x_2 \cdots x_n y$ (resp. $A^{x_1 x_2 \ldots x_n y} = B$)

(b) $x_i \in T_i$, $1 \leq i \leq n$, $y \in N(P)$

(c) $A \leq H_1$, $A^{x_1 x_2 \ldots x_i} \leq H_{i+1}$ all $1 \leq i \leq n-1$

The general form of Alperin theorem is that the family F of all pairs (H,T), where H is any Sylow intersection of the form $H = P \cap Q$ with Q also a Sylow p-subgroup of G and both $N_P(H)$ and $N_Q(H)$ are Sylow p-subgroups of N(H), T=N(H), is a conjugation family for P. These Sylow intersections H are called tame Sylow intersections. Some examples of conjugation families and weak conjugation families are also given in [1]; Among them the family $F_C = \{(H,T)\}$ where $H = P \cap Q$ is a tame intersection and T=C(H) if $C_P(H) \nleq H$ while T=N(H) if $C_P(H) \leq H$, is a conjugation family, this means that the fusion pattern of such subgroups H in N(H) with $C_P(H) \leq H$ determines the complete fusion pattern of subsets of P in G. Goldschmidt in [8] showed that the fusion is in fact determined by the p-local subgroups N(H) where N(H)/H is p-isolated.

For fusion of subsets in a p-subgroup of G, rather than a Sylow p-subgroup of G, one can find some account in Kantor and Seitz's article [16]. Also more theorems about local and global properties of finite groups can be found in the following references: Alperin and Gorenstein [4], Finkel [6], Hall [10], Glauberman [7] and Thompson [20].

Intuitively, a good conjugation family requires either fewer members in the family or more groups in the family are of known types.

A family F of local subgroups is said to control the local fusion of the Sylow p-subgroup P if :

(2.4) Whenever two elements x, y of P are conjugate in G, then there exist $N_1, N_2, \ldots, N_r$ in F and elements $g_1, g_2, \ldots, g_r$ such that $g_i \varepsilon N_i$, $i=1,2,\ldots,r$ and $x^{g_1 g_2 \cdots g_r} = y$.

We will call such a family a local fusion family, note that here in the local conjugation steps, each pair of elements is not required to lie in the p-subgroup whose normalizer in one of $N_i$.

The area of interest of this paper is to examine the fusion of elements in the Sylow p-subgroups of the symmetric group of any degree and to find some "good" local fusion families of the Sylow p-subgroups. That is, to find some families of p-subgroups of G whose normalizers control the fusion of p-elements in G. As was proved by Alperin, the normalizers of all p-subgroups of the Sylow p-subgroup P is certainly one such family, but it is much too big to be significant. The significance of this paper is its discovery of some very nice local fusion families for the Sylow p-subgroups of the symmetric group of a p-power degree. Each such family contains normalizers of p-subgroups of only two different structures. In fact, they are one "large" elementary abelian group and many "small" elementary abelian groups of order $p^2$.

In order to deal with permutations of high degrees and look into their cycle structures we have to use many new

notations which work as language of the discussion and make the theory of this paper sound.

Only knowledge of fundamental group theory such as Sylow theorems is needed. Some counting methods are also used in the proofs of the theorems. In chapter II Sylow subgroups of symmetric groups are discussed, some elementary results are given, and a complete discussion of $S_{p^n}$ (the Sylow p-subgroup of the symmetric group of degree $p^n$) as wreath products of $S_{p^{n-1}}$ and $C_p$ (cyclic group of order p) in two ways are shown. Also some group-theoretic terminologies are introduced. The main theorem is proved in two parts; the first part is proved in Chapter III and the second part is proved in Chapter IV. Some examples illustrating the proofs of these theorems are given in the appendix. All groups are considered finite in this paper.

# CHAPTER II
# SYLOW SUBGROUPS OF SYMMETRIC GROUPS

## 3. Symmetric Groups

Given any set $\Omega$ of order m, a <u>one-to-one</u> map from $\Omega$ onto itself is called a <u>permutation</u> on $\Omega$. The set of all permutations on $\Omega$ forms a group under the composition of maps, called the <u>symmetric group of degree m</u>, and denoted by $\gamma_m$. It is a simple combinatorial matter to see that $|\gamma_m|=m!$. Any subgroup of $\gamma_m$ is called a <u>permutation group of degree m</u>. Cayley's theorem (see [11] §2.9) shows that every finite group can be realized as a <u>permutation group</u>, that is, every finite group has a <u>representation</u> as a permutation group. This clearly shows that the symmetric groups themselves merit close examination.

Let $\gamma_m$ be a symmetric group on the set $\Omega$ of m elements. For each $x \varepsilon \gamma_m$, x induces a decomposition of $\Omega$ into disjoint subsets, namely, the equivalence classes of the equivalence relation: $\alpha \equiv \beta$ if and only if there is an integer i such that $\beta = \alpha x^i$. These disjoint subsets are called <u>orbits under x</u>. If $\alpha \varepsilon \Omega$ then the <u>orbit of $\alpha$ under x</u> consists of all elements $\alpha x^i$ with $i=0,\pm1,\pm2,\ldots$. For each $\alpha \varepsilon \Omega$ there is a smallest positive integer d dependent upon $\alpha$ such that $\alpha x^d = \alpha$ ; thus the orbit of $\alpha$ under x consists of exactly d elements: $\alpha, \alpha x, \alpha x^2, \ldots, \alpha x^{d-1}$. By the <u>cycle of x containing $\alpha$</u> we mean the

ordered set $(\alpha,\alpha x,\alpha x^2,\alpha x^3,\ldots,\alpha x^{d-1})$ considered as the restriction of the map x on this orbit. The integer d is called the length of the cycle. The commas "," between elements in the cycle are usually omitted. For example if $\alpha=1$, $\alpha x=5$, $\alpha x^2=7$, $\alpha x^3=3$, $\alpha x^4=1$, then the cycle of x containing $\alpha$ will be (1573). Two different cycles on the same orbit may represent the same permutation; for example, (1573), (5731), (7315), (3157) all represent the same map on the orbit $\{1,5,7,3\}$. For each permutation x and $\alpha,\beta\varepsilon\Omega$ the symbol $x:\alpha\to\beta$ means x maps $\alpha$ to $\beta$. We will also use $\alpha^x=\beta$ to mean the same thing. The following proposition concerning the permutations are elementary and its proof can be found in most algebra or group theory texts:

PROPOSITION 2.1. *Let* $\gamma_m$ *be a symmetric group on* $\Omega$ *and* $x,y,z,\varepsilon\gamma_m$ *then*

(1) *The order of x is the least common multiple of the lengths of disjoint cycles under x.*

(2) *If* $z^{-1}xz=y$, (i.e. $x^z=y$), *then x and y have the same cycle structure; that is, there is a one-to-one correspondence between orbits of x and orbits of y which preserves the length of orbits.*

(3) *If* x,y *are conjugate* in $G=\gamma_m$, (i.e. *there is* $z\varepsilon G$ *such that* $x^z=y$) *and if*

(3.1) $$x=(a_1a_2\ldots a_{n_1})(b_1b_2\ldots b_{n_2})\ldots(x_1x_2\ldots x_{n_r})$$
$$y=(\alpha_1\alpha_2\ldots\alpha_{n_1})(\beta_1\beta_2\ldots\beta_{n_2})\ldots(\chi_1\chi_2\ldots\chi_{n_r})$$

are cycle decompositions of x and y (as products of disjoint cycles), then the element of g sending $a_i$ to $\alpha_i$, $b_j$ to $\beta_j$ ..., $x_e$ to $\chi_e$ for all i, j, ...e is a conjugating element for x to y, (i.e. $x^g = y$).

## 4. Direct Products, Semidirect Products and Wreath Products

The terminologies contained in the following definition are needed:

DEFINITION 2.2. Let A, B be two groups

(a) If A, B are subgroups of a group G with $G=AB$ and $A\cap B=E$, we call the subgroup A a complement to the subgroup B in G and A,B are said to be complementary in G.

(b) By the direct product of A and B we mean a group G containing two normal complementary subgroups $A^*$, $B^*$ with $A^*\simeq A$ and $B^*\simeq B$. G is denoted by $G=A\times B$.

(c) By a semidirect product of A by B (or a split extension over A by B) we mean a group G containing a normal subgroup $A^*$ isomorphic to A and a subgroup $B^*$ isomorphic to B, such that $A^*$ and $B^*$ are complementary in G. We write $G=[A]B$ to mean G is a semidirect product of A by B.

(d) Let A be a group and B a permutation group on $\Omega$ with $|\Omega|=n$, then the wreath product $A\wr B$ of A by B is the set

(4.1) $G=\{(f,b)/f:\Omega\to A,\ b\in B\}$

with the multiplication

(4.2) $(f_1,b_1)(f_2,b_2) = (g,\ b_1b_2)$ where $g=\Omega\to A$ is defined $g(\partial)=f_1(\partial)f_2(\partial^{b_1})$ for all $\partial\in\Omega$. (Here $\partial^{b_1}$ represents the image of $\partial$ under the permutation $b_1$, while we use $g(\partial)$ to to denote the image of $\partial$ under the map g).

It can be shown the multiplication is associative and the element $(f,E)$, where $f(\partial)=E$ all $\partial\varepsilon\Omega$, is the identity of the wreath product and the inverse of the element $(f,b)$

(4.3) $\quad (f,b)^{-1}=(g,b^{-1})$ where $g(\partial)=[f(\partial b^{-1})]^{-1}$ $\partial\varepsilon\Omega$.

Hence G is a group. The group B is called the "top" group, A is called the "bottom" group of the wreath product.

PROPOSITION 2.3. Let B be a permutation group on the set $\Omega=\{1,2,3,\ldots,n\}$ and A a group. Let $G=A\wr B$ be the wreath product of A by B, then

(1) G contains a normal subgroup $N=A_1xA_2x\ldots xA_n$

(4.4) $\quad$ Where $A_i=\{(f,E)\,|\,f(\partial)=E$ for all $\partial\neq i\}\simeq A$

(2) The subgroup $B^*=\{(e,b)\,|\,b\varepsilon B,\ e(\partial)=E$ all $\partial\varepsilon\Omega\}$ is isomorphic to B and $G=NB^*$ with $N\cap B^*=E$, that is, G is the semi-direct product of N by B, i.e.

(4.5) $\quad G=[AxAx\ldots xA]B$ and hence

(4.6) $\quad |A\wr B|=|A|^n|B|$

(3) If A is also a permutation group on the set $\Gamma$, then $G=A\wr B$ is a permutation group on $\Gamma x\Omega$ with the action:

(4.7) $\quad (\gamma,\partial)(f,b)=(\gamma f(\partial),\partial b)$ for each $\gamma\varepsilon\Gamma$ and $\partial\varepsilon\Omega$ and $(f,b)\varepsilon G$. (Here, $\partial b$ means $\partial^b$, $\gamma f(\partial)$ means $\gamma^{f(\partial)}$)

(4) For any permutation groups A on $\Gamma$, B on $\Omega$ and C on $\Delta$ $(A\wr B)\wr C$ and $A\wr(B\wr C)$ are permutation groups on $(\Gamma x\Omega)x\Delta$ and $\Gamma x(\Omega x\Delta)$ respectively and

(4.8) $\quad (A\wr B)\wr C\simeq A\wr(B\wr C)$

Thus, the wreath product "$\wr$" as composition between permutation groups, is associative. We will prove only part (3), the proofs of other parts can be found in Huppert [13] Ch. 1 §15.

PROOF: (3) For each $(f,b)\varepsilon G$, $(\gamma,\partial)\varepsilon\Gamma x\Omega$ , define $(f,b)^*$ on $\Gamma x\Omega$ by

(4.9) $(\gamma,\partial)(f,b)^*=(\gamma f(\partial),\ \partial b)$ then

(i) $(f,b)^*$ so defined is a one-to-one map from $\Gamma x\Omega$ into $\Gamma x\Omega$: For let

$$(\gamma_1,\partial_1)(f,b)^*=(\gamma_2,\partial_2)(f,b)^*$$

then $(\gamma_1 f(\partial_1),\partial_1 b)=(\gamma_2 f(\partial_2),\partial_2 b)$

so $\gamma_1 f(\partial_1)=\gamma_2 f(\partial_2)$ and $\partial_1 b=\partial_2 b$

Since b, considered as a permutation on $\Omega$, is one-to-one, $\partial_1=\partial_2$. This implies $\gamma_1 f(\partial_1)=\gamma_2 f(\partial_1)$, again because $f(\partial_1)$ is a permutation on $\Gamma$ and hence one-to-one. So we have $\gamma_1=\gamma_2$. Thus $(\gamma_1,\partial_1)=(\gamma_2,\partial_2)$ and $(f,b)^*$ is a one-to-one map.

(ii) $(f,b)^*$ is onto: For each $(\partial,\gamma)\varepsilon\Gamma x\Omega$.

(4.10) Let $\partial^*=\partial b^{-1}$ and $\gamma^*=\gamma(f(\partial^*)^{-1}$ then

$$\begin{aligned}(\gamma^*,\partial^*)(f,b)^* &= (\gamma^* f(\partial^*),\ \partial^* b)\\ &= (\gamma\cdot\ (f(\partial^*))^{-1}\cdot\ f(\partial^*),\ \partial b^{-1}\cdot b)\\ &= (\gamma,\partial)\end{aligned}$$

So $(f,b)^*$ is onto map.

(iii) The map $(f,b)\to(f,b)^*$ is a homorphism: for let $(f_1,b_1)$ , $(f_2,b_2)\varepsilon$ $G=A\wr B$. We need to show

(4.11) $(f_1,b_1)^*(f_2,b_2)^* = [(f_1,b_1)(f_2,b_2)]^*$

For each $(\gamma,\partial)\varepsilon\Gamma x\Omega$

$$\begin{aligned}(\gamma,\partial)(f_1,b_1)^*(f_2,b_2)^* &= (\gamma f_1(\partial),\partial b_1)(f_2,b_2)\\ &= [(\gamma f_1(\partial)]f_2(\partial b_1),\ \partial b_1 b_2)\\ &= [\gamma g(\partial),\ \partial(b_1 b_2)]\end{aligned}$$

Here $g(\partial)=f_1(\partial)f_2(\partial^{b1})$ works as the function g defined in

$(f_1,b_1)(f_2,b_2)=(g,b_1b_2)$ in (4.2)

thus $(\gamma,\partial)\ (f_1,b_1)^*(f_2,b_2)^* = (\gamma,\partial)\ (g,b_1b_2)^*$

$= (\gamma,\partial)\ (\ (f_1,b_1)(f_2,b_2)\ )^*$

So $(f_1,b_1)^*(f_2,b_2)^* = (\ (f_1,b_1)(f_2,b_2))^*$

and $(f,b) \to (f,b)^*$ is a homorphism.

## 5. Sylow Subgroups of Symmetric Groups

Given a positive integer m and a prime p, there exist non-negative integers $a_o$, $a_1\ldots,a_r$, all less than p, such that

(5.1) $$m=a_rp^r+a_{r-1}p^{r-1}+\ldots+a_3p^3+a_2p^2+a_1p+a_o$$

It is well known [17] that the number of powers of p which appear in the product $m!= m(m-1)(m-2)\ldots3.2.1$ is

(5.2) $$e(m)=a_r(1+p+p^2+\ldots+p^{r-1})+a_{r-1}(1+p+p^2+\ldots+p^{r-2})+\ldots$$
$$+a_3(1+p+p^2)+a_2(1+p)+a_1$$

This means $|m|_p=p^{e(m)}$, if we use $|m|_p$ to denote the maximal p-power dividing m, for any integer m and prime p. Hence, the order of the Sylow p-subgroups of the symmetric group $\Upsilon m$ on a set $\Omega$ of m elements is $p^{e(m)}$.

Let C be the cyclic group of order p and let

(5.3) $\wr^n C=C\wr C\wr C\ldots\wr C$, n times for all $n=1,2,3,\ldots$ let $\wr^o C=E$. The proof of the following proposition can be found in [14] by Kaloujnine 1948:

PROPOSITION 2.4. If the cyclic group C of order p is considered as a regular permutation group on the set $\Omega=\{1,2,3,\ldots p\}$, then

(1) $\wr^n C$ is a Sylow p-subgroup of the symmetric group $\Upsilon p^n$ of degree $p^n$ on the set $\Omega_n=\Omega x\Omega x\ldots x\Omega$, n times.

(2) <u>The Sylow p-subgroups of the symmetric group $\gamma_m$ where m satisfies (5.1), are isomorphic to the</u>

(5.4) group $\underbrace{(\iota^r C \times \iota^r C \times \ldots \times \iota^r C)}_{a_r \text{ times}} \times \underbrace{(\iota^{r-1} C \times \iota^{r-1} C \times \ldots \times \iota^{r-1} C)}_{a_{r-1} \text{ times}}$

$\ldots \times \underbrace{(\iota^2 C \times \iota^2 C \times \ldots \times \iota^2 C)}_{a_2 \text{ times}} \times \underbrace{(\iota^1 C \times \iota^1 C \times \ldots \times \iota^1 C)}_{a_1 \text{ times}} \times \underbrace{(\iota^o C \times \iota^o C \times \ldots \iota^o C)}_{a_o \text{ times}}$

<u>or, in short,</u>

(5.5) $\prod_{i=o}^{r} \underbrace{(\iota^i C \times \iota^i C \times \ldots \times \iota^i C)}_{a_i \text{ times}}$ or $\prod_{i=o}^{r} (\prod_{j=1}^{ai} (\iota^i C)_j)$

This proposition shows that the sylow p-subgroups of symmetric groups of any degree are direct products of sylow p-subgroups of symmetric groups of p-power degrees. This fact motivates the close study of the sylow p-subgroups $\iota^n C$ of the symmetric group $\gamma_{p^n}$ for any positive integer n.

By the Sylow Theorems, all sylow p-subgroups of a finite group are conjugate; finding the structure and the fusion pattern of one sylow p-subgroup will tell the structure and fusion pattern of all sylow-p-subgroups. From now on, we will be studying a particular sylow p-subgroup of $\gamma_{p^n}$ on the set $\Omega_n$ of $p^n$ elements.

Let $\Omega_n$ be the vector space of dimension n over the

(5.6) field $\Omega=\{1,2,3,\ldots,p\}$

which is isomorphic to the field $\{Z_p;+,.\}$ of integers modulo p. (We will use both the symbols p and $0$ to denote the class $[0]$ in $\{Z_p;+,.\}$. We consider $\Omega_n$ as $\Omega\times\Omega\times\ldots\times\Omega$ n times. i.e.

(5.7) $\Omega_n=\{[v_1,v_2,\ldots v_n] \mid v_i \in \Omega,\ i=1,2,\ldots,n\}$ with component-wise addition and multiplication.

For each $j=1,2,3,\ldots,n-1$, $\Omega_j$ will denote the $j$-dimensional subspace $\Omega x\Omega x\ldots x\Omega x0x0x\ldots x0$

(5.8) $$\Omega_j=\{[v_1,v_2,\ldots v_j,0,0,\ldots,0] \mid v_i\varepsilon\Omega, i=1,2,\ldots,j\}$$

An arbitrary vector in $\Omega_n$ will usually be denoted by $v^{(n)}=[v_1,v_2,\ldots v_n]$, or $v^n$. Here we use the bracket [ ] to denote vectors in order to be distinguishable from the ( ) which represents cycles. If $j$ is the largest $i$ such that $v_i \neq 0$, then $v^{(n)}$ is labelled by $v^{(j)}$ (or $v^j$) and is considered as a vector in the $j$-dimensional subspace $\Omega_j$. The superscript $(j)$, which represents the dimension, will be omitted if it is clear what dimension is being discussed.

Let $\gamma_p{}^n$ be the symmetric group on the set $\Omega_n$ and let $S_p{}^n$ be the set of all permutations which are defined by

(5.9) $$x:[v_1,v_2\ldots,v_n] \rightarrow [v_1+x_o, v_2+x_1(v_1), v_3+x_2(v_1,v_2), v_4+x_3(v_1,v_2,v_3),\ldots v_n+x_{n-1}(v_1,v_2,\ldots,v_{n-1})]$$ for all $v=v^n=[v_1,v_2,\ldots,v_n]\varepsilon\Omega_n$

where $x_o\varepsilon\Omega$, $x_1=\Omega \rightarrow \Omega$, $x_2=\Omega_2 \rightarrow \Omega,\ldots,$ $x_{n-1};\Omega_{n-1} \rightarrow \Omega$ are arbitrary maps. $x$ so defined is denoted by

(5.10) $$x=\{x_o,x_1,x_2,\ldots,x_{n-1}\}$$

Kaloujnine in [15] proved that $Sp^n$ defined above is a group isomorphic to a sylow $p$-subgroup of the symmetric group $p^n$. Our permutations are written on the right i.e. the image of a vector $v$ under the permutation $x$ is written $vx$. While the functions $x_i$ in (5.9) are written on the left, e.g. $x_i(v_1,v_2\ldots v_i)$ represents the image of the vector $[v_1,v_2\ldots,v_i]\varepsilon\Omega_i$ under the map $x_i$.

## 6. Structure of $S_p n$

Let us define and examine the generating elements of $S_p n$ and some other elements and subgroups which play important roles in this paper:

For each $j<n-1$ and each $v^{(j)} \varepsilon \Omega_j$ define

$$(6.1) \qquad t_{v(j)} = \{\underbrace{0,0,\ldots,0}_{j}\ \xi_{v}(j),\ \underbrace{0,0,\ldots,0}_{n-j-1}\}$$

where $\xi_v(j) : \Omega_j \to \Omega$ is defined by

$$\xi_v(j)(w^{(j)}) = \{\begin{matrix} 1 & w^{(j)} = v^{(j)} \\ 0 & \text{otherwise} \end{matrix}$$

$t_{v(j)}$ moves only vectors with $v^{(j)}$ as their first j entries and maps $[v^{(j)}, v_{j+1}, *,*,\ldots,*]$ to $[v^{(j)}, v_{j+1}+1, *,*,\ldots,*]$ here * denotes arbitrary entries which are not related to what is being discussed. Let

$$(6.2) \qquad t_o = \{1,0,0,\ldots,0\}$$ which maps $[v_1\ldots,v_n]$ to $[v_1+1, v_2,\ldots,v_n]$ for all $[v_1\ldots,v_n] \varepsilon \Omega_n$. For each $j \leq n-2$, $v^{(j)} \varepsilon \Omega_j$, let

$$(6.3) \qquad T_{v(j)} = \text{the subgroup generated by } \{t_{[v(j),a]}\}_{a \varepsilon \Omega}$$

and

$$(6.4) \qquad t_{v}(j)^* = \prod_{a \varepsilon \Omega} t_{[v^{(j)},a]}$$ which is the generator of the diagonal subgroup of the group $T_{v(j)}$. Finally,

$$(6.5) \qquad t_o^* = \prod_{a \varepsilon \Omega} t_{[a]} = \{0,1,0,0,\ldots,0\} \, ,$$

For each $j \leq n-1$, let

$$(6.6) \qquad A^j = \prod_{v^{(j-1)} \varepsilon \Omega_{(j-1)}} T_{v(j-1)}$$ = the subgroup generated by all $t_{v(j)}$ with $v^{(j)} \varepsilon \Omega_j$. In particular, we use T to denote $A^{n-1}$ which is the subgroup generated by all $t_{v(n-1)}$'s, $v^{(n-1)}$, $\varepsilon \Omega_{n-1}$. i.e.

(6.7) $T = A^{n-1} = \langle t_{v(n-1)} \rangle_{v(n-1) \varepsilon \Omega_{n-1}}$

It is the subgroup T which plays the key role in the main result of this paper. Let us describe the above notations graphically:

Consider a "sun" sitting in the center of a "big" circular orbit of p primary planets, each planet represents a number in $\Omega$. Then consider each primary planet as sitting at the center of a secondary orbit containing p secondary planets each represents a vector $v^2$ in $\Omega_2$. For example, the planet corresponding to $v^2=[2,3]$ is the number 3 secondary planet sitting on the secondary orbit which arounds the primary planet numbered 2. See Figure 2.1:

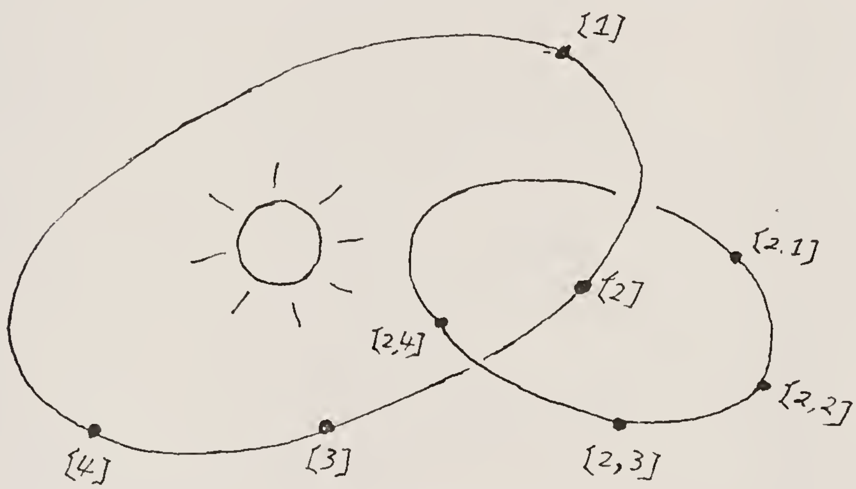


Figure 2.1

Then for each secondary planet, there is an orbit (called third order orbit) centered at it which contains p third order planets. Each represents a vector $v^3 \varepsilon \Omega_3$. Proceed this until the "nth order" planets sitting on the "nth order" orbits around the (n-1)th order planets. See Figure 2.2.

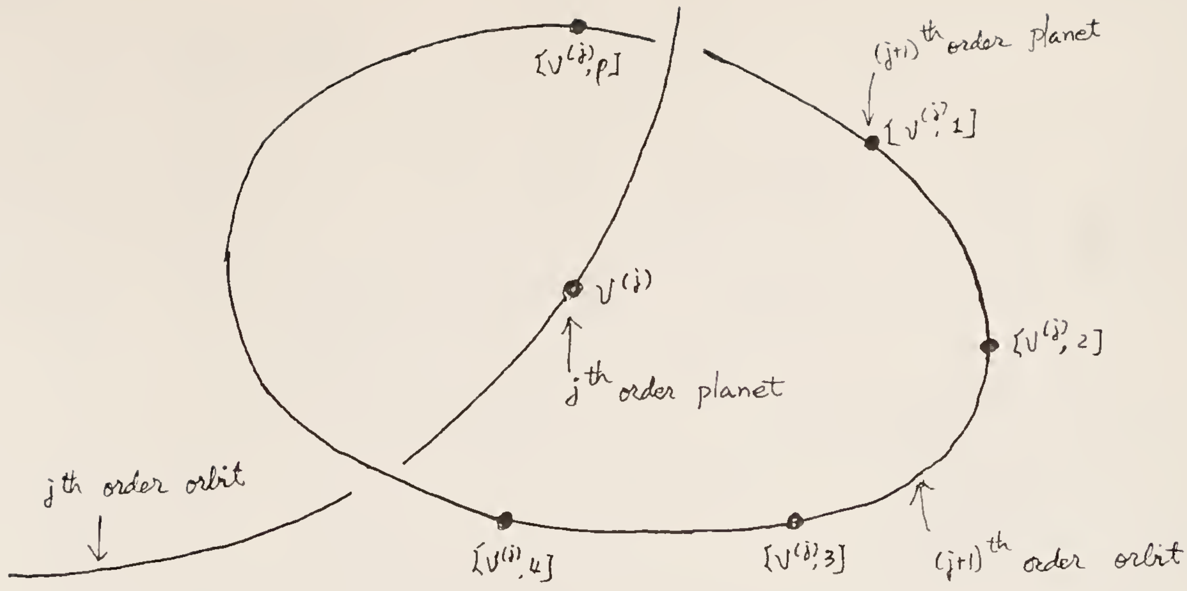


Figure 2.2

Thus, we have a "Galaxy" which contains a complex chain of orbits and planets. The element $t_o$ represents a transformation of the Galaxy obtained by a 1/p-revolution of the primary orbit. (In the same direction as the order the planets on the orbit are labelled.)

For each $a \varepsilon \Omega$, $t_{[a]}$ represents the transformation obtained by a 1/p-revolution of the secondary orbit around the primary planet corresponding to [a]. In general, for each $j \leq n-1$ and $v^{(j)} \varepsilon \Omega_j$, $t_{v^{(j)}}$ represents the transformation obtained by a 1/p-revolution of the (j+1)th order orbit around the jth order planet corresponding to the vector $v^{(j)}$. $t_o^*$ represents the transformation obtained by rotating all p secondary orbits 1/p-revolution simultaneously. In general, for each $v^{(j)} \varepsilon \Omega_j$, $t^*_{v(j)}$ represents the transformation obtained by rotating 1/p-revolution simultaneously all p (j+2)th order orbits which are around the p (j+1)th order planets sitting on

the (j+1)th orbit centered at the jth order planet which corresponds to the vector $v^j$. The subgroup of T is the group of transformations obtained by any combination of rotations of all $p^{p^{n-1}}$ nth order orbits, each a multiple of 1/p-revolution. We have

PROPOSITION 2.5. (1) T is an elementary abelian group of order $p^{p^{n-1}}$.

(2) All t's and t*'s are of order p.

(3) $A^{(j)}$ is an elementary abelian group of order $p^{p^j}$.

(4) Each $T_{v^{(j)}}$ is an elementary group of order $p^p$.

PROOF: Trivial.

PROPOSITION 2.6. (1) If $i \le j$ and $v^i \varepsilon \Omega_i$, $w^j \varepsilon \Omega_j$, then $t_{v^i}$ and $t_{w^j}$ commute if and only if i=j or $v^i$ doesn't coincide with the first i entries of $w^j$.

(2) If i<j and $v^i \varepsilon \Omega_i$, $w^j \varepsilon \Omega_j$ such that $v^i$ coincides with the first i entries of $w^j$, i.e.

(6.8) If $w^j = [v^i, w_{i+1}, w_{i+2}, \ldots, w_j]$, then

(6.9) $(t_{w^j})^{t_{v^i}} = t_v$ where $v = w^j + 1^j_{i+1} \varepsilon \Omega_j$

(6.10) with $1^j_{i+1} = [0,0,\ldots 0,\underset{\substack{\uparrow \\ (i+1)\text{th entry}}}{1},0,0,\ldots,0]$

PROOF (1) We first prove the "if" part:

(a) Assume i=j and $v^i \ne w^j$. Since $t_{v^i}$ fixes all vectors except those containing $v^i$ as first i entries and $t_{w^j}$ fixes all vectors except those containing $w^j$ as the first j (=i) entries, every vector is either fixed by $t_{v^i}$ or $t_{w^j}$. Thus, $t_{v^i} t_{w^j} = t_{w^j} t_{v^i}$.

(b) Assume i<j and $v^i$ doesn't coincide with the first i entries of $w^j$. Then with the same reason as in (a),

every vector is either fixed by $v^i$ or by $w^j$; hence $t_{v^i}$ and $t_{w^j}$ commute, The "only if" part of (1) will be shown after we prove part (2):

(2) Assume $i<j$ and $v^i$ coincides with first i entries of $w^i$ as in (6.8). Since $t_{w^j}$: $[v^i, w_{i+1}\ldots,w_j,w_{j+1},*,*,\ldots,*]\to$ $[v^i,w_{i+1},\ldots,w_j,w_{j+1}+1,*,*,\ldots,*]$ (vectors not mentioned are fixed). By part (3) of proposition 2.1,

$$(t_{w^j})^{t_{v^i}}: [v^i,w_{i+1},\ldots,w_j,w_{j+1},*,*,\ldots,*]t_{v^i}\to$$
$$(v^i,w_{i+1},\ldots,w_j,w_{j+1},+1,*,*,\ldots,*]t_{v^i}.$$

this is : $[v^i,w_{i+1}+1,w_{i+2},\ldots,w_j,w_{j+1},*,*,\ldots,*]\to$
$$[v^i,w_{i+1}+1,w_{i+2},\ldots,w_j,w_{j+1},+1,*,*,\ldots*]$$

Which is the same action as $t_v$ where

$v = w^j+1^j_{i+1}$; thus (2) is proved.

For the "only if" part of (1), suppose $i\neq j$ and $v^i$ coincides with the first i entries of $w^j$; then by what we just proved: $(t_{w^j})^{t_{v^i}} \neq t_{w^j}$. This implies $t_{w^j}\, t_{v^i} \neq t_{v^i}\, t_{w^j}$; the proposition is proved.

The structure of the sylow p-subgroup $S_{p^n}$ is analyzed in the next proposition (also see Weir [21]):

PROPOSITION 2.7. (1) <u>$S_{p^n}$ is generated by all t's defined in (6.1) and (6.2), consequently,</u>

(6.11) $S_{p^n}=A^0A^1A^2\ldots A^{n-1}$ where $A^0=\langle to\rangle$

(2) <u>$S_{p^n}$ is a semidirect product of the group T by $S_{p^{n-1}}$,</u> i.e.

(6.12) $S_{p^n}=[T]\ S_{p^{n-1}}$. <u>Furthermore, there is a linear equivalence relation on $S_{p^n}$ such that the equivalence classes have a group structure isomorphic to $S_{p^{n-1}}$.</u>

(3) <u>If $C_p$ is the cyclic group of order p, then</u>

(6.13) $S_{p^n} = C_p \wr S_{p^{n-1}}$ <u>and also</u>

(6.14) $S_{p^n} = S_{p^{n-1}} \wr C_p$

PROOF Let $x \in S_{p^n}$ and $x=\{x_0,x_1,x_2,\ldots,x_{n-1}\}$ as defined in (5.9)

(1) For each $i=1,2,3,\ldots,n-1$ and each $[v_1,v_2,\ldots,v_i] \in \Omega_i$ the product

(6.15) $$\prod_{v_i=1}^{p} (t_{[v_1,v_2,\ldots,v_{i-1},v_i]})^{x_i[v_1,v_2,\ldots v_{i-1},v_i]}$$

is independent of the order of the multiplications in the $v_i$ product by part (1) of proposition 2.6. Here $x_i[v_1,\ldots,v_i] \in \Omega$ is considered an integer modulo p, since the order of the element $t_{[v_1,\ldots,v_i]}$ is p. Thus the power $(t_{[\ ]})^{x_i(\ )}$ is well-defined. Consider the element

(6.16) $$x^{\#} = \prod_{v_1=1}^{p} ( \prod_{v_2=1}^{p} (\ldots(\prod_{v_{n-1}=1}^{p} ((\ldots(t_{[v_1,v_2,\ldots v_{n-1}]}^{x_{n-1}(v_1,v_2,\ldots v_{n-1})}$$

$$t_{[v_1,\ldots,v_{n-2}]}^{x_{n-2}[v_1,\ldots,v_{n-2}]})\ldots)$$

$$t_{[v_1,v_2]}^{x_2[v_1,v_2]})\ t_{[v_1]}^{x_1(v_1)})\ t_0^{x_0}$$

in which the products are executed in the order that $\prod_{v_{n-1}=1}^{p}$ first, then followed by $\prod_{v_{n-2}=1}^{p}$, and so on until $\prod_{v_1=1}^{p}$ then finally multiplied by $t_0^{x_0}$.

We assert that $x^{\#} = x(=\{x_0,x_1,\ldots x_{n-1}\})$, for each vector $W = [w_1,w_2,\ldots,w_n]$, by 5.9. $w\,x=[w_1+x_0,w_2+x_1(w_1), w_3+x_2(w_1,w_2),\ldots,w_n+x_n(w_1,w_2,\ldots,w_{n-1})]$. On the other hand, $w = [w_1,\ldots,w_n]$ is fixed by all t's except $t_0$,

$t_{[w_1]}, t_{[w_1,w_2]}, \ldots t_{[w_1,\ldots,w_{n-1}]}$; hence the image of w by $x^{\#}$ is the same as its image by the element.

$$(6.17) \qquad t_{[w_1,\ldots w_{n-1}]}^{x_{n-1}[w_1,\ldots w_{n-1}]} \; t_{[w_1,\ldots w_{n-2}]}^{x_{n-2}[w_1,\ldots w_{n-2}]} \ldots t_{[w_1,w_2]}^{x_2[w_1,w_2]} \; t_{[w_1]}^{x_1[w_1]} \; t_0^{x_0}$$

which obviously maps $w = [w_1,\ldots,w_n]$ to the element $[w_1+x_0, w_2+x_1(w_1),\ldots,w_n+x_{n-1}(w_1,w_2,\ldots,w_{n-1})]$; hence $w\,x = w\,x^{\#}$; thus $x=x^{\#}$. This proves that $S_pn$ is generated by all t's. Since $A^i$ is generated by all $t_v(i)$'s with $v^{(i)}\varepsilon\Omega$, so $S_pn$ factors into $A^0A^1A^2,\ldots A^{n-1}$.

(2) First we show that $T \trianglelefteq S_pn$; for each generator $t_{v(n-1)}$ of T and every $x\varepsilon S_pn$ by (1) x factors into $x=\partial_0\partial_1,\ldots,\partial_{n-1}$ with $\partial_i\varepsilon A^i$, $i=0,1,2,\ldots,n-1$, so $t_{v(n-1)}^{\;x} = t_{v(n-1)}^{\partial_0\partial_1,\ldots\partial_{n-1}} = t_{v(n-1)}^{\partial_0\partial_1,\ldots\partial_{n-1}} = t_{v(n-1)}^{\partial_0}\, t_{v(n-1)}^{\partial_1}\ldots t_{v(n-1)}^{\partial_{n-1}}$.

Since each $\partial_i$ is a product of $t_v(i)$'s with $v^{(i)}\varepsilon\Omega i$ by part 1 and 2 of proposition 2.6, $t_{v(n-1)}^{t_v(i)}$ either equals $t_{v(n-1)}$ (in which part 1 is applied) or equals $t_v\varepsilon T$ for some $v\varepsilon\Omega_{n-1}$ (in which part 2 is applied). Thus, $t_{v(n-1)}^{\partial i}\varepsilon T$ all i hence $T\trianglelefteq S_pn$. Second, we show that $A^0A^1,\ldots A^{n-2}$ is a subgroup of $S_pn$ which is isomorphic to $S_pn-1$ and is complementary to T in $S_pn$. The following lemma will show that $A^0A^1,\ldots A^{n-2}$ is a subgroup of $S_pn$:

LEMMA 2.7.1 If G is a group with subgroups S,T such that $S\leq N_G(T)$, then ST=TS is a subgroup of G.

PROOF It is trivial, hence, omitted.

Now as a consequence of proposition 2.6 $A^0 \leq N(A^1)$, $A^0A^1 \subseteq N(A_2), \ldots, A^0A^1A^2, \ldots A^{n-3} \subseteq N(A^{n-2})$; thus $A^0A^1$, $A^0A^1A^2, \ldots A^0A^1A^2, \ldots A^{n-2}$ are all subgroups of $S_pn$. For each $i=1,2,\ldots n-1$, by the definition of $A^i$ (6.6) the subgroup $A^0A^1,,,A^i$ is generated by all $t_{v^i}$'s where $v^i$is an arbitrary element in $\Omega_j$ which means that $A^0A^1,\ldots A^i$ is isomorphic to the sylow p-subgroup $S_pi+1$ of the symmetric group $\gamma_pi+1$ on $\Omega_{i+1}$, on the convention that only the first i+1 entries of every vector of $\Omega_n$ are "moved" by elements in $A^0A^1,\ldots A^i$. Thus,

(6.18) $A^0A^1,\ldots A^i \simeq S_pi+1$ for each $i=1,2,3,\ldots,n-1$.

In particular

$A^0A^1A^2,\ldots A^{n-2} \simeq S_{pn-1}$ and hence

(6.19) $S_{pn} \simeq [T]\ S_{pn-1}$. (It is obvious that $T \cap A^0A^1,\ldots A^{n-2} = E$)

Finally, if we define a relation "$\sim$" on $S_pn$ by:

(6.20) For each x, $y \varepsilon S_pn$, define $x \sim y$ if and only if x and y induce the same permutation on $\Omega_{n-1}$. (Here $\Omega_{n-1}$ can be considered to be the set of all "blocks" of the form $v^{(n-1)} \times \Omega \leq \Omega_n$.) Now (5.9) shows that the action of x on the first n-1 entries of vectors is independent of its action on the nth entry. (Hence, $x \sim x$ for every $x \varepsilon S_pn$.) It is obvious that "$\sim$" is an equivalence relation on $S_pn$ with the linear property

(6.21) If $x_1 \sim y_1$ , $x_2 \sim y_2$ then $x_1x_2 \sim y_1y_2$

The equivalence classes $S_pn/\sim$ form a group (called

the quotient group of the equivalence relation) with the multiplication $[x][y]=[xy]$. We conclude part (2) of this proposition by proving the following lemma:

LEMMA 2.7.2 <u>The map</u> $\theta$: $S_pn/T \to S_pn/\sim$ <u>defined by</u> $\theta(xT) = [x]$ <u>is an isomorphism</u>.

PROOF (of the Lemma): (i) $\theta$ is well-defined: for if $x_1T = x_2T$ with $t\varepsilon T$ such that $x_1=tx_2$, then for each $v^{(n)} = [v^{(n-1)},a]\varepsilon\Omega_n$;
$[v^{(n-1)},a]x_1 = [v^{(n-1)},a]tx_2 = [v^{(n-1)},a']x_2$ where $a'$ depends on a,t and $v^{(n-1)}$. Thus, $x_1$ and $x_2$ induce the same permutation on all $v^{(n-1)}$ of $\Omega_{n-1}$, thus $x_1 \sim x_2$.
$[x_1]=[x_2]$ in $S_pn/\sim$.

(ii) $\theta$ is one-to-one: if $[x_1]=[x_2]$ in $S_pn/\sim$ then $x_1 \sim x_2$, and $x_1$ and $x_2$ induce the same permutation on each $v^{(n-1)}\varepsilon\Omega_{n-1}$; for each $a^*\varepsilon\Omega$ and $w^{(n-1)}\varepsilon\Omega_{n-1}$ let $[w^{(n-1)},a^*]x_1^{-1} = [v^{(n-1)},a]$; then $[v^{(n-1)},a]x_1 = [w^{(n-1)},a^*]$. Since $[v^{(n-1)},a]x_1$ and $[v^{(n-1)},a]x_2$ have the same first n-1 entries. So let $[v^{(n-1)},a]x_2 = [w^{(n-1)},a^{**}]$; then $t=x_1^{-1}x_2 : [w^{(n-1)},a^*] \to [w^{(n-1)},a^{**}]$ hence $t\varepsilon T$; thus $x_1T=x_2T$. $\Theta$ is one-to-one.

(iii) $\Theta$ is obviously an <u>onto</u> map.

(iv) $\Theta(xT.yT) = \Theta(xyT) = [xy] = [x][y] = \theta(xT)\cdot\theta(yT)$

(i)-(iv) prove $\Theta$ is an isomorphism so the proof of (2) is completed. The image of any $x\varepsilon S_pn$ in $S_pn/T \simeq S_pn-1$ is denoted by $\bar{x}$, i.e. $x=\{x_0,x_1,\ldots x_{n-1}\}$; then $\bar{x}=\{x_0,x_1,\ldots x_{n-2}\}$.

(3) By (2) $S_pn=[T]S_pn-1$ and $T=\langle t_{v(n-1)}\rangle_{v(n-1)\varepsilon\Omega_{n-1}} = \prod_{v(n-1)\varepsilon\Omega_{n-1}} \langle t_{v(n-1)}\rangle$. There are $p^{n-1}$ vectors $v^{(n-1)}$;

hence, there are $p^{n-1}$ direct summands in the direct product T, each isomorphic to $C_p$, the cyclic group of order p; thus $S_{p^n}$ can be considered to be the wreath product $C_p \wr S_{p^{n-1}}$ which is the group of all pairs [see (4.1)] $\{(f,b); f:\Omega_{n-1} \to C_p, b \in S_{p^{n-1}}\}$ in the following way: If $x=\{x_0,x_1,\ldots,x_{n-1}\}$ let the pair (f,b) be $b=\bar{x}=\{x_0,x_1,\ldots,x_{n-2}\}$ and $f[(v_1,\ldots,v_{n-1})]=x_{n-1}(v_1,v_2,\ldots,v_{n-1})$; that is, each element $x=\{x_0,x_1,\ldots,x_{n-1}\}$ corresponds to the pair $(x_{n-1},\bar{x})$ as defined in (4.1) and (4.2). Hence, for each vector

(6.22) $\quad v^{(n)}=[v^{(n-1)},v_n] \in \Omega_n$ , $[v^{(n-1)},v_n](x_{n-1},\bar{x})=[v^{(n-1)}\bar{x}, v_n+x_{n-1}(v^{(n-1)})]$. If $y=\{y_0,y_1,\ldots,y_{n-1}\}$ then

(6.23) $\quad [v^{(n-1)},v_n](v_{n-1},\bar{x})(y_{n-1},\bar{y})= [v^{(n-1)}\bar{x},v_n+x_{n-1}(v^{n-1})](y_{n-1},\bar{y}) = [v^{(n-1)}\bar{x}\bar{y}, v_n+x_{n-1}(v^{(n-1)})+y_{n-1}(v^{(n-1)}\bar{x})]$

On the other hand, the composition given in (4.2) is

(6.24) $\quad (x_{n-1},\bar{x})(y_{n-1},\bar{y})= (g,\bar{x}\bar{y})$ where $g = \Omega_{n-1} \to C_p$ is defined by $g(v^{(n-1)}) = x_{n-1}(v^{(n-1)})y_{n-1}(v^{(n-1)}\bar{x})$ and hence, its action on $v^{(n)}=[v^{(n-1)},v_n]$ is

(6.25) $\quad [v^{(n-1)},v_n][g,\overline{xy}] = [v^{(n-1)}\overline{xy},v_n+g(v^{(n-1)})]$

$\quad\quad = [v^{(n-1)}\bar{x}\bar{y},v_n+x_{n-1}(v^{(n-1})+y_{n-1}(v^{(n-1)}\bar{x})]$

(6.23) and (6.25) are the same; this proves that the correspondence between x's and the pairs $(x_{n-1},\bar{x})$ defined above is an isomorphism. Hence, $S_{p^n} \cong C_p \wr S_{p^{n-1}}$.

Notice that it was proved that $S_{p^n} \cong C_p \wr C_p \wr \ldots \wr C_p$, n times, which can be viewed either as $C_p \wr (C_p \wr \ldots \wr C_p)$ or as $(C_p \wr \ldots \wr C_p) \wr C_p$; the induction gives $(C_p \wr \ldots \wr C_p)$, n-1 times, as the group $S_{p^{n-1}}$; thus $S_{p^n}$ is isomorphic to both

$C_p \wr S_{pn-1}$ and $S_{pn-1} \wr C_p$. We have shown how $S_{pn}$ is isomorphic to $C_p \wr S_{pn-1}$; we are now going to see how $S_{pn}$ is viewed as the wreath product $S_{pn-1} \wr C_p$:

Consider $S_{pn-1}$ as the permutation group on the set $\Omega^*_{n-1} = \{[0,v_2,\ldots,v_n] \mid v_i \in \Omega\,,\ i = 2,3,\ldots,n\} \leq \Omega_n$ and the cyclic group Cp as a permutation group acting on $\Omega^* = \{[v_1,0,\ldots,0] \mid v_i \in \Omega\} \leq \Omega_n$. Here we use $\Omega^*_{n-1}$ and $\Omega^*$ instead of $\Omega_{n-1}$ and $\Omega$ to remind the reader what entries of vectors they act on in order to illustrate the definition of wreath product given in (4.1), (4.2) and (4.7).

For each $x = \{x_0, x_1, \ldots, x_{n-1}\}$, let the corresponding pair (f,b) in the definition of wreath product $S_{pn-1} \wr C_p$ in (4.1) and (4.2) be defined by:

(6.26) $\quad$ b is a permutation on $\Omega_*$ and $b: v_1 \to v_1 + x_0$

$f: \Omega^* \to S_{pn-2}$ is given by $f(v_1): \Omega^*_{n-1} \to \Omega^*_{n-1}$ and

(6.27) $\quad f(v_1): [v_2, v_3, \ldots, v_n] \to [v_2 + x_1(v_1), v_3 + x_2(v_1, v_2), \ldots,$
$v_n + x_n + (v_1, v_2, \ldots, v_{n-1})]$.

It is a simple algebraic matter to check that with the definitions (6.26) and (6.27), the composition of pairs satisfies (4.2) and (4.7); hence, $S_{pn}$ is isomorphic to the wreath product $S_{pn-1} \wr C_p$.

## 7. Normalizers And Centralizers

In this last section of Chapter II we will discuss the structures of the normalizers and centralizers of the two particular p-subgroups of $S_{pn}$, which control the local fusion. First, we look at the "big" elementary abelian group T:

PROPOSITION 2.8 <u>Let T be defined as in (6.7) let $G=\gamma_{p^n}$, then</u>

(7.1) $N_G(T) \simeq ([Cp]Cp-1) \wr \gamma_{p^{n-1}}$

(7.2) $N_{S_{p^n}}(T)\ ([Cp]Cp-1) \wr S_{p^{n-1}}$

(7.3) $C_G(T) \simeq T$ <u>where $C_p$ and $C_{p-1}$ are cyclic groups of order p and p-1 respectively</u>.

PROOF First note that if $g \varepsilon N(T)$, then g induces a permutation $\bar{g}$ on $\Omega_{n-1}$ which is the action of g on the first n-1 entries of the vectors. For if there were vectors $[v^{(n-1)},a]$ and $[v^{(n-1)},a+\ell]$ such that

$$[v^{(n-1)},a]g = [w_1{}^{(n-1)},a^*] \text{ and}$$

$$[v^{(n-1)},a+\ell]g = [w_2{}^{(n-1)},a^{**}] \text{ with } w_1^{(n-1)} \neq w_2{}^{(n-1)}$$

then since $(t_{v^{(n-1)}})^{\ell}$: $[v^{(n-1)},a] \to [v^{(n-1)},a+\ell]g$

$$((t_{v^{(n-1)}})^{\ell})^{g} \quad : [v^{(n-1)},a]g \to [v^{(n-1)},a+\ell]g$$

$$: [w_1{}^{(n-1)},a^*] \to [w_2{}^{(n-1)},a^*]$$

because $w_1{}^{(n-1)} \neq w_2{}^{(n-1)}$ $(t_{v^{(n-1)}})^{\ell})^{g} \not\in T$. This contradicts the assumption that $G \varepsilon N(T)$.

The conjugation by g of elements in T is an automorphism of T; thus it maps a generating set to a generating set. Since T is generated by all $t_{v^{(n-1)}}$ with $v^{(n-1)}$ ranging over all vectors in $\Omega_{n-1}$, the conjugation by g permutes the $p^{n-1}$ cyclic subgroups $t_{v^{(n-1)}}$ . Since the set of permutations on these $p^{n-1}$ cyclic subgroups is isomorphic to $\gamma_{p^{n-1}}$, it constitutes the "top" group of the wreath product. For the notation of <u>top group</u> see definition 2.2. Now let $\beta \varepsilon \gamma_{p^{n-1}}$ be the element of the top group

corresponding to g; for each $v=v^{(n-1)}\in\Omega_{n-1}$, let $w=w^{(n-1)}$ be the vector in $\Omega_{n-1}$ such that $(<t_v>)^g=<w>$, that is, $v\beta=w$. Now we have the p-cycles $t_v=t_v^{(n-1)}$ = $([v,1]\ [v,2]\ldots[v,p])$ and $t_w=t_w(n-1)$ = $([w,1]\ [w,2]\ldots[w,p])$. The bottom group of the wreath product contains the set of maps.

(7.4) $f:\{[v,1], [v,2],\ldots,[v,p]\}\rightarrow\{[w,1],[w,2],\ldots,[w,p]\}$ satisfying

(7.5) $(t_v)^{f_a}=(t_w)^a$ for some $a\in\{1,2,3,\ldots,p-1\}$.

Here $f_a$ is one such map which corresponds to $a\in\Omega$. Then in the cycles $t_v=t_v(n-1)=([v,1][v,2],\ldots,[v,p])$ and $(t_w)^a=([w,1][w,1+a][w,1+2a],\ldots,[w,1+^{(p+1)}a])$. $f_a$'s must match the vectors in their order but free of choice for the first vector in the cycle. Thus, for each $b\in\{0,1,2,\ldots,p\}$there corresponds a map $f_a^b$ which maps $[v,v_n]$ to $[w,1+(b+v_n-1)a]$ for all $v_n=1,2,\ldots,p$; i.e.

$f_a^b$: $[v,v_n]\rightarrow[w,v_n.a+(ab-a+1)]$. For each fixed a, $\{ab-a+1\}_{b\in\Omega}$ is the same set as $\Omega$ itself. ($b\rightarrow ab-a+1$ is a linear function of b). Thus, as b ranges over all elements in $\Omega$, $f_a^b$ ranges over all maps of the form

(7.6) $f_a^b$: $[v,v_n]\rightarrow[w,v_n.a+b]$.

The set of all such maps $f_a^b$ can be realized as the group of <u>all Affine transformations</u> on the <u>Affine line.</u> This group is the semidirect product of $C_p$ by $C_{p-1}$ where $C_p$ is the set of all $f^b$'s and $C_{p-1}$ is the set of all $f_a$'s. Now this group $[C_p]C_{p-1}$ is the "bottom" group of the wreath product because the above definition of $[C_p]C_{p-1}$

comes from every $v=v^{(n-1)} \varepsilon \Omega_{n-1}$ and the pairs $(f,\beta)$ with $\beta \varepsilon \gamma_{p^{n-1}}$ and $f= \Omega_{n-1} \to [cp]Cp-1$ is $f: v^{(n-1)} \to f^b_a$ defined above satisfy (4.1) and (4.2) of the definition of wreath product. Hence, $N_G(T) \simeq ([Cp]Cp-1) \wr \gamma_{p^{n-1}}$.

It is easy to see (7.2) comes from (7.1), to show (7.3). Let g be an element in C(T); then $t_{v^{(n-1)}}^g = t_{v^{(n-1)}}$ for all $v^{(n-1)} \varepsilon \Omega_{n-1}$. For each $v^{(n-1)}$ and all i=1,2,...,p, the permutations:

$$Z^i_{v^{(n-1)}}: [v^{(n-1)},v_n] \to [v^{(n-1)},v_n+i] \text{ all } v_n \varepsilon \Omega$$

all satisfy $t_{v^{(n-1)}}^{Z^i_{v^{(n-1)}}} = t_{v^{(n-1)}}$ and every permutation on $<v^{(n-1)}>$ which satisfy this equation comes from one such $Z^i_{v^{(n-1)}}$; thus the group $\prod_{v^{(n-1)} \varepsilon \Omega_{n-1}} <Z_{v^{(n-1)}}>$, which is isomorphic to T, is the centralizer of T in G. The proposition is proved.

Recall that for each $v^{(n-2)} (=v) \varepsilon \Omega_{n-2}$

(7.7) $\quad t^*_v = \prod_{a \varepsilon \Omega} ([v,a,1][v,a,2],\ldots[v,a,p])$ is a product of p p-cycles and

(7.8) $\quad t_v = \prod_{b \varepsilon \Omega} ([v,1,b][v,2,b],\ldots[v,p,b])$ is also a product of p p-cycles. We now discuss the structures of the normalizer and centralizer of the group generated by $t^*_v$ and $t_v$:

PROPOSITION 2.9 <u>For each $v=v^{(n-2)} \varepsilon \Omega_{n-2}$</u>

(7.9) $\quad N(<t_v,t^*_v>) \simeq ([Cp]Cp-1) \times ([Cp]Cp-1) \rtimes \gamma_\ell$

(7.10) $\quad C(<t_v,t^*_v>) \simeq (CpxCp) \times \gamma_\ell$ <u>where $\gamma_\ell$ is the symmetric group of degree $p^n-p^2$ on the set $\Omega_n-\{v\}x\Omega_2$.</u>

PROOF It is obvious that every permutation of $\Omega_n$

leaving all $p^2$ vectors in $\{v\}\times\Omega_2$ fixed lies in the normalizer and centralizer of $\langle t_v, t_v^*\rangle$. Hence, the factor $\gamma_\ell$ has little significance as far as the structure is concerned. Thus, through the proof of this proposition we will use $t$ to denote $t_v$ and use $t^*$ to denote $t_v^*$ and $[a,b]$ to denote $[v,a,b]$; hence, we are looking at the symmetric group $\gamma_{p^2}$ on the set $\Omega_2$. We prove (7.10) first:

(7.11) $\quad t^* = \prod_{b\in\Omega} ([a,1][a,2],\ldots,[a,p])$

(7.12) $\quad t = \prod_{b\in\Omega}([1,b][2,b],\ldots,[p,b])$

If $g\in C(\langle t,t^*\rangle)$ then $t^g=t$ and $t^{*g}=t^*$. From $t^{*g}=t^*$, $g$ permutes the $p$ $p$-cycles $\{[a,1][a,2],\ldots,[a,p]\}$; hence, $g$ induces a permutation $\bar{g}_1$ on the first entries; from $t^g=t$, $g$ permutes the $p$ $p$-cycles $\{[1,b][2,b],\ldots,[p,b]\}$; hence, $g$ induces a permutation $\bar{g}_2$ on the second entries; thus, $g:[a,b]\to[a\bar{g}_1,b\bar{g}_2]$ all $a,b$. Hence,

(7.13) $\quad ([a,1][a,2],\ldots,[a,p])^g=([a\bar{g}_1,1\bar{g}_2[[a\bar{g},2\bar{g}_2],\ldots,$
$[a\bar{g},p\bar{g}_2]) = ([a\bar{g},1][a\bar{g}_1,2][a\bar{g}_1,2],\ldots,[a\bar{g}_1,p])$. Hence, there is $j$ such that the corresponding

(7.14) $\quad g_j:[a,b]\to[a\bar{g}_1,b+j]$ all $a,b$, likewise,
$([1,b][2,b],\ldots,[p,b])^g = ([1\bar{g}_1,b\bar{g}_2][1\bar{g}_1,b\bar{g}_2],\ldots,[p\bar{g}_1,b\bar{g}_2]$
$= ([1,b\bar{g}_2][2,b\bar{g}_2],\ldots,[p,b\bar{g}_2])$.

Hence, there is $i$ such that the corresponding map

(7.15) $\quad g^i;[a,b]\to[a+i,b\bar{g}_2]$. Hence, for each pair $i,j$, there corresponds a map $g_j^i:[a,b]\to[a+i,b+j]$. These $g_j^i$'s are all the permutations in the centralizer of $\langle t,t^*\rangle$. The set of all $g_j^i$'s, where $i,j=0,1,2,\ldots,p-1$, can be realized as the direct product $C_p \times C_p$ which is isomorphic to $\langle t^*,t\rangle$ thus $C(\langle t^*,t\rangle) = \langle t^*,t\rangle \simeq C_p\times C_p$.

Next we show $N(<t^*,t>) \simeq ([Cp]Cp-1)\times([Cp]Cp-1)$.
If $g \varepsilon N(<t^*,t>)$ then there exist $i,j \varepsilon \{0,1,2,..,p-1\}$ such that

(7.16) $\quad (t^*)^g = t^{*j}t^i : [a,b] \to [a+i,b+j]$ all $a,b \varepsilon \Omega$.

We will find all g such that 7.16 holds and $t^g \varepsilon <t^*,t>$, since $t^* = \prod_{a\varepsilon\Omega} ([a,1][a,2],...,[a,p])$

$(t^*)^g = \prod_{a\varepsilon\Omega} ([a+i,1+j][a+i,2+j],...,[a+i,p+j])$; thus g can be found to send the cycle $([a,1][a,2],...,[a,p])$ to any of the p p-cycles in $(t^*)^g$. Let $\partial \varepsilon \gamma_p$ be a permutation on $\Omega$ which sends the vectors in the cycle $([a,1],...,[a,p])$ to the cycle $([a^\partial+i,1+j][a^\partial+i,2+j],...,[a^\partial+i,p+j])$.
Let $g_\partial$ be the element of $Sp^2$ which corresponds to such $\partial$.
Now, for each $\partial$, and each a, there are p ways to match the vectors $\{[a,1],[a,2],...,[a,p]\}$ and the vectors $\{[a^\partial+i,1+j],[a^\partial+i,2+j],...,[a^\partial+i,p+j]\}$; i.e. there exists $a^\ell$ depending on a such that

(7.17) $\quad g_\partial^\ell : [a,b] \to [a^\partial +i,b+j+a^\ell]$

Since $t = \prod_{b\varepsilon\Omega} ([1,b][2,b],...,[p,b])$ and $t^g$ must lie in $<t^*,t>$, say $t^g = t^{*j'}t^{i'}$ for some integer $i',j'$ modulo p, then

(7.18) $\quad t^g: [a,b] \to [a+i',b+j']$ which coincides with $\prod_{b\varepsilon\Omega} ([1^\partial+i,b+j+1^\ell](2^\partial+i,b+j+2^\ell],...,[p^\partial+i,p+j+p^\ell])$ which reads as $[a^\partial+i,b+j+a^\ell] \to [(a+1)^\partial+i,b+j+(a+1)^\ell]$. Comparing (7.18) and (7.19) we have

(7.20) $\quad (a+1)^\partial - a^\partial = i'$ and

(7.21) $\quad (a+1)^\ell - a^\ell = j'$

Thus (7.20) and (7.21) are necessary conditions which define the elements $g \varepsilon N(<t^*,t>)$; these limit the ways

of assigning the cycles and in each cycle, the ways of "shifting" the vectors. Note that the definition (7.17) of $g^{\ell}_{\partial}$ doesn't depend on i,j because as a,b range over all $\Omega$, the pair $[a^{\partial}+i, b+j+a^{\ell}]$ also ranges over all $\Omega_2$. Hence, the definitions of $\partial$ and $\ell$ (both permutations on $\Omega$) define the element $g \varepsilon N(\langle t^*, t\rangle)$. It is easy to see the definition of $\partial$ and $\ell$ satisfying (7.20) and (7.21) is equivalent to the definition of the direct product $([Cp]Cp-1)x([Cp]Cp-1)$ of pairs $(\partial,\ell)$ where $\partial:\Omega \rightarrow \Omega$, $\ell:\Omega \rightarrow \Omega$ such that $\partial(a)=ha+m$ all $a\varepsilon\Omega$ and $\ell(a)=Ka+m'$ all $a\varepsilon\Omega$ where $h,K \varepsilon \{1,2,3,\ldots,p-1\}$ and $m,m' \varepsilon\{0,1,2,\ldots,p\}$. Hence, $N(\langle t^*,t\rangle) \simeq ([Cp]Cp-1)x([Cp]Cp-1)$ the proof of proposition 2.9 is completed.

# CHAPTER III
# FUSION OF ELEMENTS
# IN $S_p n$ I

## 8. Cycle Structure and Conjugacy Classes

Each element x of $S_p n$ can be written as a product of disjoint cycles. If we consider vectors fixed by x as cycles of length $1=p^\circ$, then all cycles which appear in the cycle decomposition of x are p-power cycles: the maximum cycle length is the order of x. Since $|\Omega_n| = p^n$ the order of x is at most $p^n$ and the sum of the lengths of all cycles (including $p^\circ$-cycles) equals $p^n$. Thus, in $S_p n$, each partition of $p^n$ into p-power numbers gives a <u>cycle decomposition</u>, and since two elements of $S_p n$ are conjugate if and only if they have the same <u>cycle decomposition</u>, each cycle decomposition represents a conjugacy class in $S_p n$. Thus, there are as many conjugacy classes in $S_p n$ as the number of partitions of the integer $p^n$ into p-power integers.

If $x=\{x_0,x_1,x_2,\ldots,x_{n-1}\}\varepsilon S_p n$ and if $x_0\neq 0$, then x fixes no vectors of $\Omega_n$. If $x_o=0$ then x is a product of p elements $\Pi_1,\Pi_2,\ldots,\Pi_p$ where for each i, $\Pi_i$ is a permutation acting on the subset $\{i\} \times \Omega_{n-1}$ of $\Omega_n$, hence, an element of $S_p n-1$. If $x\varepsilon T$, then $x_0,x_1,x_2,\ldots x_{n-2}$ are all zero maps and x is a product of as many p-cycles as the number of vectors $v^{(n-1)}$ such that $x_{n-1}(v^{(n-1)})\neq 0$.

For each $v^{(j)} \varepsilon \Omega_j$, $j=1,2,\ldots,n-1$, $t_{v(j)}$ and $t^*_{v(j)}$ are both products of $p^{n-j-1}$ p-cycles, while $t_0$ and $t_o{}^*$ are products of $p^{n-1}$ p-cycles.

## 9. The Main Theorem

For each $x \varepsilon S_{p^n}$, let $\bar{x}$ be the image of x under the canonical homomorphism: $S_{p^n} \to S_{p^n}/T \simeq S_{p^{n-1}}$; i.e. $\bar{x}$ is a permutation of degree $p^{n-1}$ induced by x on the first n-1 entries of the vectors of $\Omega_n$. $\bar{x}$ can be called the action of x on the "blocks" which are subsets of $\Omega_n$ of the form $\{v^{(n-1)}\} x \Omega$ where $v^{(n-1)}$ ranges over $\Omega_{n-1}$. Define $\bar{\bar{x}}$ as the permutation on $\Omega_{n-2}$ induced by x in the same way.

Although there are a large number of conjugacy classes, the fusion of elements in $S_{p^n}$ is suprisingly simple; as we will prove soon, only two kinds of local subgroups are needed to control the local fusion of $S_{p^n}$. They are the normalizer of the big elementary abelian group T, and the normalizers of some small elementary abelian groups of order $p^2$. This is stated in the following theorem:

MAIN THEOREM: <u>The following two families of local groups control the local fusion of $S_{p^n}$</u> :

(9.1) $\quad F_1 = \{N(T), N(\langle t_v{}^*(n-1), t_{v(n-1)}\rangle)$; $v^{(n-2)}$ ranges over $\Omega_{n-2}\}$

(9.2) $\quad F_2 = \{N(T), N(\langle \prod_v t^*_v, \prod_v t_v \rangle)$; where the products range over any subset (the same set for both products) of $\Omega_{n-2}$.$\}$

We will prove the main theorem in two parts: Theorem 1

proves the fusion of elements x and y in $S_p n$ with x and y conjugate and $\bar{x}$ and $\bar{y}$ are also conjugate. Theorem 2 proves the general case.

THEOREM 1: If $x,y \varepsilon S_p n$ are conjugate and if $\bar{x}$ and $\bar{y}$ are also conjugate, then there is an element g in the normalizer of T such that $x^g=y$. In other words, N(T) controls the local fusion of elements in $S_p n$ having the same cycle structure on "blocks".

PROOF Let $x = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$ and $y = \{y_0, y_1, y_2, \ldots, y_{n-1}\}$ as defined in (5.9). For each $v \varepsilon \Omega_n$, let $\{vx\}$ denote the orbit of v under x; thus $\{vx\}$ is the set of all vectors in the cycle of x containing the vector v. Let $|\{vx\}|$ denote the cycle length. For each $v^{(n-1)} \varepsilon \Omega_{n-1}$ and each $v^{(n-2)} \varepsilon \Omega_{n-2}$, the orbits $\{v^{(n-1)}\bar{x}\}$ and $\{v^{(n-2)}\bar{\bar{x}}\}$ and their cycle lengths are defined in the same way. First, we have to prove a lemma concerning the relation between the cycle lengths of vx and $v^{(n-1)}\bar{x}$ when $v=[v^{(n-1)},a]$ for some $a\varepsilon\Omega$ :

LEMMA 3.1 Let $x \varepsilon S_p n$, for each $v^{(n-1)} \varepsilon \Omega_{n-1}$ and each $a \varepsilon \Omega$, if $|\{v^{(n-1)}\bar{x}\}| = p^d$ for $d \geq o$, then either

$|\{[v^{(n-1)},a]x| = p^d$ or

$|\{[v^{(n-1)},a]x| = p^{d+1}$.

Furthermore, if $|\{[v^{(n-1)},a]x| = p^\gamma$ for one $a\varepsilon\Omega$ where $\gamma \geq o$, then $|\{[v^{(n-1)},b]x| = p^\gamma$ for all $b\varepsilon\Omega$.

PROOF Assume $|\{v^{(n-1)}\bar{x}\}| = p^d$. Define

$$(9.3) \qquad \delta x(v^{(n-1)}) = \sum_{w^{(n-1)} \varepsilon \{v^{(n-1)}\bar{x}\}} \chi_{n-1}(w^{(n-1)})$$

For each $a\varepsilon\Omega$

$[v^{(n-1)},a]x = [v^{(n-1)}\bar{x},\ a+x_{n-1}\ (v^{(n-1)})]$

$(v^{(n-1)},a]x^2 = [v^{(n-1)}\bar{x}^2, a+x_{n-1}(v^{(n-1)})+x_{n-1}(v^{(n-1)}\bar{x})]$

. . .

(9.4) $[v^{(n-1)},a]x^{p^d} = [v^{(n-1)},\ a+\delta x(v^{(n-1)})]$ because of $v^{(n-1)}\bar{x}^{p^d} = v^{(n-1)}$ and the definition of $\delta x(v^{(n-1)})$.

Case (i) If $\delta x(v^{(n-1)}) \equiv 0$ (modulo p), then for any $a\varepsilon\Omega$, $[v^{(n-1)},a]x^{p^d} = [v^{(n-1)},a]$ by (9.4) and $p^d$ is the smallest power of x which fixes the vector $[v^{(n-1)},a]$. Hence, $|\{[v^{(n-1)},a]x\}| = p^d$ and in the orbit $\{[v^{(n-1)},a]x\}$, all the vectors $v^{(n-1)}$'s are different so there is no $b \neq a$ in $\Omega$ such that $[v^{(n-1)},b]\varepsilon\ \{[v^{(n-1)},a]x\}$. So, $\{[v^{(n-1)},b]x\}$ $b=1,2,\ldots,p$ are p disjoint orbits of length $p^d$. We call a $p^d$-cycle of x of this type a type I $p^d$-cycle of x.

Case (ii) If $\delta_x(v^{(n-1)}) \not\equiv 0$ (modulo p), then

$[v^{(n-1)},a]x^{p^d} = [v^{(n-1)},a+\delta x(v^{(n-1)})]$ by (9.4)

$[v^{(n-1)},a]x^{2p^d} = [v^{(n-1)},a+2\cdot\delta x(v^{(n-1)})]$

. . .

(9.5) $[v^{(n-1)},a]x^{ip^d} = [v^{(n-1)},a+i\cdot\delta x(v^{(n-1)})]$

Since $a+i\cdot\delta x(v^{(n-1)})$ ranges over all vectors in $\Omega$ as i ranges over $\Omega$, $\{[v^{(n-1)},a+i\cdot\delta x(v^{(n-1)})]\}_{i=1}^{p}$ is equal to $\{[v^{(n-1)},a]\}_{a\varepsilon\Omega}$. Hence, $[v^{(n-1)},a]x^{p^{d+1}} = [v^{(n-1)},a]$ and $x^{p^{d+1}}$ is the least power of x which fixes $[v^{(n-1)},a]$. Hence, $|\{[v^{(n-1)},a]x\}| = p^{d+1}$ and the orbit $\{[v^{(n-1)},a]x\}$ contains $[v^{(n-1)},b]$ for all $b\varepsilon\Omega$. Hence, $|\{[v^{(n-1)},b]x| = p^{d+1}$ for all $b\varepsilon\Omega$. We call $p^{d+1}$-cycles of x of this type a type II $p^{d+1}$-cycle of x. Thus, the lemma is proved.

Now since x and y are conjugate in $\gamma_{p^n}$, their cycle structures are the same. That is, they have the same number of $p^r$-cycles for every $r=0,1,2,\ldots,n$. Furthermore, since $\bar{x}$ and $\bar{y}$ are conjugate in $\gamma_{p^{n-1}}$, $\bar{x}$ and $\bar{y}$ also have the same number of $p^r$-cycles for every $r=0,1,2,\ldots,n-1$. The following lemma is the key to the proof of this theorem:

LEMMA 3.2 <u>If $x,y \in S_{p^n}$ are conjugate and if $\bar{x}$ and $\bar{y}$ are also conjugate, then for each $r=0,1,2,\ldots,n$ the number of type I $p^r$-cycles of x equals the number of type I $p^r$-cycles of y, and the number of type II $p^r$-cycles of x and y are equal.</u>

PROOF We prove the lemma by induction. For each $r=0,1,2,\ldots,n$ let $X_1^r$ (resp. $X_2^r$) be the number of type I (resp. type II) $p^r$-cycles of x. Let $Y_1^r$ (resp. $Y_2^r$) be the number of type I (resp. type II). $p^r$-cycles of y and let $X^r, Y^r$ be the number of $p^r$-cycles of x and y respectively. (Here $X^r=Y^r$ because x and y are conjugate.) Then, $X^r=X_1{}^r+X_2{}^r$, $Y^r=Y_1{}^r+Y_2{}^r$, for $r=0$, $X_1{}^0, Y_1{}^0$, are numbers of fixed points of x and y respectively and $X_2{}^0=Y_2{}^0=0$ because there is no type II $p^0$-cycle. So, obviously $X_1{}^0=Y_1{}^0=X^0=Y^0$. Let's now look further at the case $r=1$; $X_2{}^1$ = the number of type I p-cycles of x which look like $([v^{(n-1)},a_1][v^{(n-1)},a_2],\ldots[v^{(n-1)},a_p])$ for some $v^{(n-1)} \in \Omega_{n-1}$ and $\{a_1,\ldots,a_p\}=\Omega$. Each such $v^{(n-1)}$ has the property that $x_{n-1}(v^{(n-1)}) \not\equiv 0$ (modulo p); thus

(9.6) $\quad X_2^1$ = the number of fixed points $v^{(n-1)}$ of $\bar{x}$ with $x_{n-1}(v^{(n-1)}) \not\equiv 0$ (modulo p).

On the other hand, if $w^{(n-1)} \varepsilon \Omega_{n-1}$ is a fixed point of $\bar{x}$ with $x_{n-1}(w^{(n-1)}) \equiv 0$ (modulo p), then $[w^{(n-1)}, a]$ is a fixed point of x for every $a \varepsilon \Omega$. This means each such $w^{(n-1)}$ gives p fixed points of x. Since there are $(\bar{X}^o - X_2^1)$ such $w^{(n-1)}$, where $\bar{x}^o$ is the number of fixed points of $\bar{x}$, we have the

(9.7) relation: $p.(\bar{X}^o - X_2^1) = X^o =$ the number of fixed points of x, the same equation holds for y;

(9.8) $p.(\bar{Y}^o - Y_2^1) = Y^o =$ the number of fixed points of y.

Now by the assumption, $X^o = Y^0$, $\bar{X}^o = \bar{Y}^o$, the above two equations give the equality

$$X_2^1 = Y_2^1 \quad ; \text{ hence, } \quad X_1^1 = Y_1^1$$

This proves the case for r=1.

Now assume the lemma holds for r=d. That is

(9.9) $X_1{}^d = Y_1{}^d$, $X_2{}^d = Y_2{}^d$.

Since for each r, $X^r = Y^r$, $\bar{X}^r = \bar{Y}^r$, we use $C^r$ to denote $X^r, Y^r$, and use $\bar{C}^r$ to denote $\bar{X}^r, \bar{Y}^r$ just to note they are constants because x and y are given to be conjugate and so are $\bar{x}$ and $\bar{y}$.

If $\{v^{(n-1)}\bar{x}\}$ is an orbit of $\bar{x}$ of length $p^d$ and if $\delta x(v^{(n-1)}) \equiv 0$ (mod p), then $v^{(n-1)}$ gives exactly p type I $p^d$-cycles of x; they are (use $v = v^{(n-1)}$) $([v,a][v\bar{x}, a + x_{n-1}(v)] \ldots, [v\bar{x}^{p^d-1}, a + {*}{*}{*}, \ldots])$ $a = 1, 2, 3, \ldots, p$.

All type I $p^d$-cycles of x arise in such a manner,

(9.10) let $\bar{m}_1^d$ be the number of $p^d$-cycles of $\bar{x}$ which give type I $p^d$-cycles of x, then

(9.11) $x_1^d = p\cdot\bar{m}_1^d$ . Now

(9.12) $\bar{C}^d - \bar{m}_1^d$ = the number of $p^d$-cycles of $\bar{x}$ which give type II $p^{d+1}$-cycles of x. Since every type II $p^{d+1}$-cycle of x comes from exactly one $p^d$-cycle of $\bar{x}$ and every such $p^d$-cycle of $\bar{x}$ gives exactly one type II $p^{d+1}$-cycle of x, hence, we have

(9.13) $\bar{C}^d - \bar{m}_1^d = X_2^{d+1}$

Combining (9.11) and (9.13) we have

(9.14) $\bar{C}^d - \frac{X_1^d}{p} = X_2^{d+1}$

the same equation holds for y; we have

(9.15) $\bar{C}^d - \frac{Y_1^d}{p} = Y_2^{d+1}$

From (9.14) and (9.15) together with the inductive hypothesis $X_1^d = Y_1^d$, we have

(9.16) $X_2^{d+1} = Y_2^{d+1}$ hence, also

(9.17) $X_1^{d+1} = X_1^{d+1}$

Thus, by induction, $X_1^r = Y_1^r$ and $X_2^r = Y_2^r$ for all r=1,2,3,... The lemma is proved.

Now for each d=0,1,2,...,n and each $p^d$-cycle of x, we can find a corresponding $p^d$-cycle of y of the same type because of Lemma 3.2. So there is a one-to-one correspondence between all cycles of x and all cycles of y preserving the length and the type. If we choose a permutation g which maps vectors of each cycle of x into vectors in the corresponding cycle of y in such a way that the order the vectors appear in each cycle is preserved; that is, if the corresponding cycles of x and y are written:

(9.18) $(v,vx,vx^2,\ldots,v_x{}^{p^d-1})$ and $(w,wy,wy^2,\ldots,wy^{p^d-1})$

then g maps vector $vx^i$ to $wx^i$ for every $i=p,1,2,\ldots,p^d-1$, then obviously $x^g=y$ if such g is defined. The final step is to show that some such g defined by an appropriate expressions of cycles of x and their corresponding cycles of y lies in the normalizer of T.

For each $v=v^{(n-1)}\varepsilon\Omega_{n-1}$, let $|\{v\bar{x}\}|=p^d$, we consider two cases at length

Case 1: Suppose $\{v\bar{x}\}$ gives type I $p^d$-cycles of x. Let the corresponding $p^d$-cycle of y be given by the orbit $\{w\bar{y}\}$ of $\bar{y}$ where $w=w^{(n-1)}\varepsilon\,\Omega_{n-1}$. Because of Lemma 3.1, the p vectors $[v,a]$ $a=1,2,\ldots,p$ all lie in different orbits of x and the p vectors $[w,a]$, $a=1,2,\ldots,p$ also lie in different orbits of y. We now choose the one-to-one correspondence such that the orbit $\{[v,a]x\}$ corresponds to the orbit $\{[w,a]y\}$ for each $a=1,2,\ldots,p$. Then we define the element g to send $[v,a]x^i$ to $[w,a]y^i$ for all $a=1,2,\ldots,p$ and all $i=0,1,2,\ldots,p^d-1$, this way the element g is <u>not</u> defined <u>only</u> on the vectors $[v,a]$ $a=1,2,\ldots,p$ <u>but also</u> on the vectors $[w,a]x^i$ for all $i=o,1,2,\ldots,p^d-1$. In other words, there are p $p^d$-cycles of x and y involved in this definition of the element g. We will now show that for each $v\bar{x}^i$ of these $p^d$ vectors $\{v\bar{x}^i\}_{i=o}^{p^d-1}$, we have $t_{v\bar{x}^i}\varepsilon T$.

For every vector $v\bar{x}^i$ in the orbit $\{v\bar{x}\}$ of $\bar{x}$, the vector given by $v\bar{x}^i$ which lies in the orbit $\{[v,a]x\}$ of x is

(9.19) $\quad [v\bar{x}^{i}, a + \sum_{j=o}^{i-1} x_{n-1}(v\bar{x}^{j})]$ and the corresponding vector of $y$ is

(9.20) $\quad [w\bar{y}^{i}, a + \sum_{j=o}^{i-1} y_{n-1}(w\bar{y}^{j})]$ let

(9.21) $\quad S_i = \sum_{j=o}^{i-1} x_{n-1}(v\bar{x}^{j})$ and $\mu_i = \sum_{j=o}^{i-1} y_{n-1}(w\bar{y}^{j})$

since $t_{v\bar{x}^i} : [v\bar{x}^{i}, a+s_i] \rightarrow [v\bar{x}^{i}, a+1+s_i]$

$\quad (t_{v\bar{x}^i})^g : [v\bar{x}^{i}, a+s_i]g \rightarrow [v\bar{x}^{i}, a+1+s_i]g$

(9.22) $\quad : [w\bar{y}^{i}, a+\mu_i] \rightarrow [w\bar{y}^{i}, (a+1)+\mu_i]$

Since $a+\mu_i$ ranges over $\Omega$ as a ranges over $\Omega$, (9.22) means $(t_{v\bar{x}^i})^g = t_{w\bar{y}^i} \varepsilon T$.

Since type I $p^d$-cycles of $x$ all come from some $p^d$-cycles of $\bar{x}$ and each such $p^d$-cycle of $\bar{x}$ gives $p$ type I $p^d$-cycles of $x$, the number of type I $p^d$-cycles is a multiple of $p$ for each $d=0,1,2,\ldots,n$. So the above defining process of the element $g$ is repeated until all type I cycles are exhausted. So, $(t_{v^{(n-1)}})^g \varepsilon T$ for all such $v^{(n+1)} \varepsilon \Omega_{n-1}$. Thus, we have defined $g$ on all $v^{(n-1)} \varepsilon \Omega_{n-1}$ which lie in some type I $p^d$-cycle of $x$ and have shown $(t_{v^{(n-1)}})^g \varepsilon T$ for all such $v^{(n-1)}$; the next step is to define $g$ on all other vectors $v^{(n-1)}$, that is, vectors $v^{(n-1)}$ which lie in some type II $p^{d+1}$-cycle of $x$.

Case 2: Suppose $\{v\bar{x}\}$ gives a type II $p^{d+1}$-cycle of $x$; then $\delta_x = \delta_x(v) \not\equiv 0$ (modulo $p$). Let $\{w\bar{y}\}$ be the orbit of $\bar{y}$ which gives the $p^{d+1}$-cycle of $y$ which corresponds to the type II $p^{d+1}$-cycle of $x$ containing $\{v\bar{x}\}$. And let $\delta_y = \delta_y(w) \not\equiv 0$ (mod $p$). Note that the orbit $\{[v,a]x\}$ contains all $p$ vectors of the form $[v,b]$ $b=1,2,3,\ldots,p$.

So does the orbit $\{[w,a]y\}$ contain all p vectors of the form $[w,b], b=1,2,\ldots,p$. Let's write these two (type II) $p^{d+1}$-cycles so that $[v,1]$ and $[w,1]$ are both the first vectors in the cycles. Then define g on these $p^{d+1}$ vectors by sending each vector in the cycle $\{[v,1]x\}$ to the corresponding vector in the cycle $\{[w,1]y\}$. That is

(9.23) $\quad g:[v\bar{x}^i, 1+s_i] \to [w\bar{y}^i, 1+\mu_i], i=0,1,2,\ldots,p^{d+1}-1.$

Let $q=p^d$; then for each $i\varepsilon\{0,1,2,\ldots,qp-1\}$ there exist $\ell,m\varepsilon\{0,1,2,\ldots,p-1\}$ such that

(9.24) $\quad i=\ell q+m.$

Since $|\{v\bar{x}\}|=p^d=q$, $v\bar{x}^i=v\bar{x}^{\ell q+m} = v\bar{x}^m$ and hence,

(9.25) $\quad S_i=\sum_{j=o}^{i-1} x_{n-1}(v\bar{x}^j) = \ell.\delta_x + S_m$ by the definition of $\delta_x$ in (9.3). Hence,

(9.26) $\quad [v\bar{x}^i, 1+S_i] = [v\bar{x}^m, 1+\ell.\delta_x+S_m]$

likewise

(9.27) $\quad w\bar{y}^i = w\bar{y}^m$ and $\mu_i=\ell.\delta_y + \mu_m$ so,

(9.28) $\quad [w\bar{y}^i, 1+\mu_i]=[w\bar{y}^m, 1+\ell.\delta_y+\mu_m]$ and

(9.29) $\quad g:[v\bar{x}^m, 1+\ell.\delta_x+S_m] \to [w\bar{y}^m, 1+\ell.\delta_y+\mu_m]$ for all $m,\ell$ .

For each fixed $m=0,1,2,\ldots,p-1$. Consider the p vectors $\{[v\bar{x}^i, 1+s_i] \mid i=\ell q+m, \ell=0,1,2,\ldots,p-1\}$ because of (9.26). This set equals $\{[v\bar{x}^m, 1+\ell.\delta_x+S_m] \mid \ell=0,1,2,\ldots p-1\}$ and the corresponding vectors in y are $\{[w\bar{y}^m, 1+\ell.\delta_y+\mu_m] \mid \ell=0,1,2,\ldots,p-1\}$

(9.30) $\quad g:[v\bar{x}^m, 1+\ell.\delta_x+S_m] \to [w\bar{y}^m, 1+\ell.\delta_y+\mu_m]$ for all $\ell=0,1,2,\ldots p-1$. And because

$t_{v\bar{x}^m}: [v\bar{x}^m, 1+\ell, \delta_x+S_m] \to [v\bar{x}^m, 2+\ell, \delta_x+S_m]$

$(t_{v\bar{x}^m})^{\delta x}: [v\bar{x}^m, 1+\ell . \delta_x+S_m] \to [v\bar{x}^m, 1+\delta_x+ \ell . \delta_x+S_m]$

$((t_{v\bar{x}^m})^{\delta_x})^g: [v\bar{x}^m, 1+\ell . \delta_x+S_m]^g \to [v\bar{x}^m, 1+(\ell+1)\delta_x+S_m]^g$

that is, $((t_{v\bar{x}^m})^{\delta_x})^g : [w\bar{y}^m, 1+\ell . \delta_y+\mu_m] \to$ $[w\bar{y}^m, 1+(\ell+1)\delta_y+\mu_m]$ because of (9.3) for all $\ell=0,1,2,\ldots,p-1$.

Since $(1+(\ell+1)\delta_y+\mu_m)-(1+\ell\delta_y+\mu_m)= \delta_y$ and since $a=1+\ell . \delta_y+\mu_m$ ranges over all elements of $\Omega$ as $\ell$ ranges over $0,1,2,\ldots,p-1$. This means $((t_{v\bar{x}^m})^{\delta_x})^g : [w\bar{y}^m, a] \to$ $[w\bar{y}^m, a+\delta_y]$ which is the same action as $(t_{w\bar{y}^m})^{\delta y}$; hence we have

(9.31) $\quad ((t_{v\bar{x}^m})^{\delta x})^g = (t_{w\bar{y}^m})^{\delta_y}$. This will imply $(t_{v\bar{x}^m})^g \varepsilon T$ with the help of the following lemma:

LEMMA 3.3 If the integers $h,k \not\equiv 0$ (modulo p) exist such that

$$((t_v)^h)^g = t_w^k \quad \underline{\text{where}}\ v,w,\varepsilon\Omega_{n-1}\ \underline{\text{and}}$$

$g\varepsilon\gamma_{pn}$, <u>then</u> $(t_v)^g \varepsilon$ T.

PROOF As h and p are relative prime there are integers $e,f\not\equiv 0$ (modulo p) such that $he + pf = 1$

$$\begin{aligned} t_v^g = (t_v^1)^g &= (t_v^{he+pf})^g = t_v^{heg} . t_v^{pfg} \\ &= t_v^{heg} . E \text{ (because } t_v^p = E) \\ &= (((t_v)^h)^g)^e . E \\ &= (t_w^k)^e \\ &= t_w^{ke} \varepsilon T \end{aligned}$$

In fact, $ke = \frac{k}{h}$ because $he\equiv 1$ (modulo p); the lemma is proved. Hence, the proof of Theorem 1 is completed.

We conclude the chapter by showing the following proposition which we will need in Chapter 4:

PROPOSITION 3.4 *If $x, y \varepsilon S_{p^n}$ such that for each $d=0.1.2,\ldots,n$, the number of type I $p^d$-cycle of $x$ and $y$ are the same and the number of type II $p^d$-cycles of $x$ and $y$ are the same, then $x$ is conjugate to $y$ in $\gamma_{p^n}$ and $\bar{x}$ is conjugate to $\bar{y}$ in $\gamma_{p^{n-1}}$.*

PROOF Equations (9.14) and (9.15) would become

(9.32) $$\bar{X}^d - \frac{X_1{}^d}{p} = X_2{}^{d+1} \quad \text{and}$$

(9.33) $$\bar{Y}^d - \frac{Y_1{}^d}{p} = Y_2{}^{d+1}$$

if we did not assume $\bar{X}^d = \bar{Y}^d$. Now, the assumptions $X_1{}^d = Y_1{}^d$ and $X_2{}^{d+1} = Y_x{}^{d+1}$ for all $d$ imply $\bar{X}^d = \bar{Y}^d$ by (9.32) and (9.33) and it is trivial that $X^d = Y^d$. So $x$ and $y$ have the same cycle structure and hence, conjugate. Also, $\bar{x}$ and $\bar{y}$ have the same cycle structure, hence conjugate in $S_{p^{n-1}}$; the proposition is proved.

# CHAPTER IV
# FUSION OF ELEMENTS IN $S_{p^n}$ II

In this chapter we will prove the second half of the main theorem, namely find the local fusion family which controls the fusion of the entire $S_{p^n}$ including conjugate elements with different cycle structures on "blocks". We will see also how fusion of direct product is related to the fusion of its direct summands, thus, see in general the fusion of elements in the Sylow p-subgroups of symmetric groups of any degree.

## 10. The Second Theorem

Theorem 1 shows that the conjugation of two elements with the same cycle structure on "blocks" can be taken from the normalizer of the subgroup T; this means N(T) provides all local fusion for elements which lie in the same "subclass" of a conjugacy class; here a subclass is a subset of a conjugacy class containing all elements in this conjugacy class which have the same cycle structure on blocks. More precisely, for each $x \in S_{p^n}$, let [x] be the conjugacy class of $S_{p^n}$ containing x; i.e. $[x]=\{y \in S_{p^n} | x^g=y$ for some $g \in \gamma_{p^n}\}$ ; and let $[\bar{x}]=\{y \in S_{p^n} | \bar{x}$ and $\bar{y}$ are conjugate in $\gamma_{p^{n-1}}\}$ (note here $[\bar{x}]$ is <u>not</u> the conjugacy class of

$S_{p^{n-1}}$). Theorem 1 shows that N(T) gives the local fusion of all elements in a subclass $[x] \cap [\bar{x}]$. See figure 4.1.

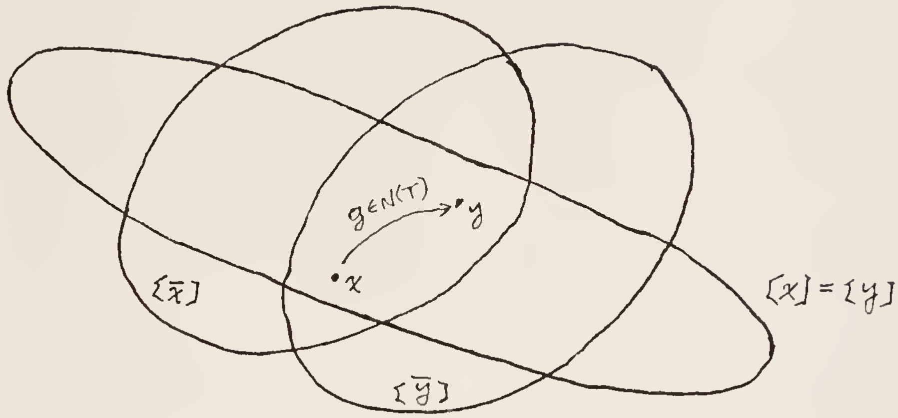


Figure 4.1

The remaining of the main theorem is to find a local conjugation family (or families) which control(s) the fusion of elements in (possibly) different subclasses. See Figure 4.2.

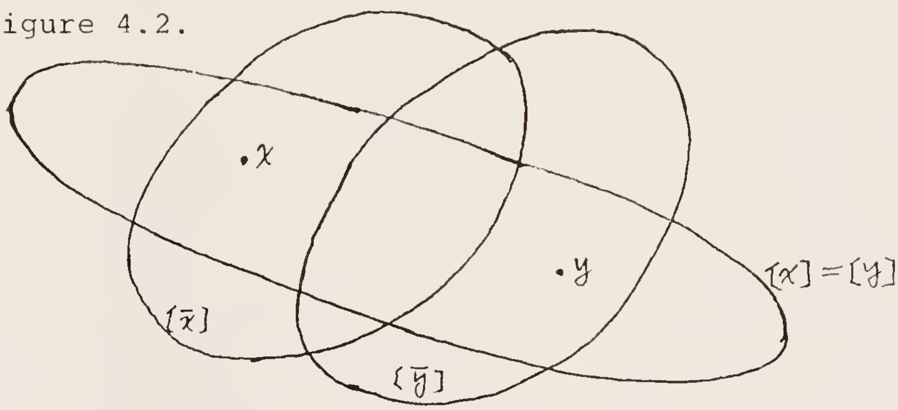


Figure 4.2

In Theorem 2, given such elements x,y we will find elements $x^*, y^*$ in the same conjugacy class such that $\bar{x}$ is conjugate to $\bar{x}^*$ and $\bar{y}$ is conjugate to $\bar{y}^*$ (thus by Theorem 1, both conjugations are in N(T)), and that $x^*, y^*$ so found depend only upon the number of cycles of all lengths and types and prove that $x^*$ is locally conjugate to $y^*$ in the families $F_1$ and $F_2$ of (9.1) and (9.2):

THEOREM 2. <u>If $x,y \in S_{p^n}$ are conjugate then there exist $g_1, g_2, \ldots, g_r$ such that $x^{g_1 g_2 \cdots g_r} = y$ and all $g_i$ lie in members of each of the following two families:</u>

$F_1 = \{N(T)$ ; $N(<t_{v^{(n-2)}}, t_{v^{(n-2)}}^*>)$ : $v^{(n-1)}$ ranges over $\Omega_{n-2}\}$

$F_2 = \{N(T)$ ; $N(< \prod_v t_v^* , \prod_v t_v>)$ : where the products range over any subset (the same set for both products) of $\Omega_{n-2}\}$

PROOF First we prove some lemmas which are needed:

LEMMA 4.1. For each $d=1,2,3,\ldots,n$ and any vector $w=w^{(n-d-1)} \in \Omega_{n-d-1}$, let $\Delta = \{w\} \times \Omega_{d-1} \leq \Omega_{n-2}$, then there exists a $p^{d-1}$-cycle $z \in S_{p^{n-2}}$ such that all vectors of $\Omega_{n-2}$ outside $\Delta$ are fixed by $z$.

PROOF (of Lemma 4.1): Let $z = \{z_0, z_1, \ldots, z_{n-3}\}$ with $z_0 = 0$ and $z_1, z_2, \ldots, z_{n-d-2}$ are zero maps, $z_{n-d-1} = \xi_w$, $z_{n-d} = \xi_{[w,p]}$, $z_{n-d+1} = \xi_{[w,p,p]}, \ldots, z_{n-3} = \xi_{[w,\underbrace{p,p,\ldots,p}_{d-2}]}$, where $\xi$'s are defined in (6.1) then it is easy to see that $z$ is a $p^{d-1}$-cycle which leaves all vectors outside $\Delta$ fixed.

LEMMA 4.2. Let $\Delta$ and $z$ be defined as in lemma 4.1 and let $v$ be any vector in $\Delta$, let $r=p^{d-1}$, define the elements $\tilde{z}_1$ and $\tilde{z}_2$ as follows:

(10.1) $\tilde{z}_1 = \prod_{b=1}^{p} ([v,1,b][vz,1,b]\ldots[vz^{r-1},1,b][v,2,b][vz,2,b]\ldots[vz^{r-1},2,b]\ldots[v,p,b][vz,p,b]\ldots[vz^{r-1},p,b])$

(10.2) $\tilde{z}_2 = \prod_{a=1}^{p} ([v,a,1][vz,a,1]\ldots[vz^{r-1},a,1][v,a,2][vz,a,2]\ldots[vz^{r-1},a,2]\ldots[v,a,p][vz,a,p]\ldots[vz^{r-1},a,p])$

then (1) $\tilde{z}_1$ is a product of p type I $p^d$-cycles while $\tilde{z}_2$ is a product of p type II $p^d$-cycles on $\Delta x \Omega x \Omega$.

(2) Let $g \varepsilon \gamma_{pn}$ such that g fixes all vectors outside $\Delta x \Omega x \Omega$ and in $\Delta x \Omega x \Omega$ let

(10.3) $\quad g:[vz^i,a,b] \to [vz^i,b,a]$ for all $i=0,1,2,\ldots,$ $r=p^{d-1}$, and all $a \varepsilon \Omega$, $b \varepsilon \Omega$.

(10.4) $\quad$ then (a) $\tilde{z}_1{}^g = \tilde{z}_2$ , $\tilde{z}_2{}^g = \tilde{z}_1$ (hence $g^2 \varepsilon C(<\tilde{z}_1,\tilde{z}_2>)$

(10.5) $\quad$ (b) $g \varepsilon N(< \prod_{i=1}^{r} t_{vz^i}$ , $\prod_{i=1}^{r} t_{vz^i}^{*}>)$

(c) g factors into $g=g_1 g_2 \ldots g_r$ such that

(10.6) $\quad g_i \varepsilon N(<t_{vz^i}$ , $t_{vz^i}^{*}>)$ for each $i=1,2,\ldots,p^{d-1}=r$,

PROOF (of lemma 4.2): (1) Since $|\{[v,1]\ \tilde{\tilde{z}}_1\}| = p^d$ and $|\{[v,1,b]\ \tilde{z}_1\}| = p^d$ for every b, thus each b gives a $p^d$-cycle of $\tilde{z}_1$ which is type I. So $\tilde{z}_1$ is a product of p type I $p^d$-cycles. While for each $a=1,2,\ldots,p$, $|\{[v,a]\tilde{\tilde{z}}_w\}| = p^{d-1}$ but $|\{[v,a,1]\tilde{z}_2\}| = p^d$, thus each a gives a type II $p^d$-cycle of $\tilde{z}_2$; hence $\tilde{z}_2$ is a product of p type II $p^d$-cycle.

(2) By the definition of the element g and part (3) of proposition 2.1, $\tilde{z}_1{}^g = \tilde{z}_2$ and $\tilde{z}_2{}^g = \tilde{z}_1$; hence $(\tilde{z}_1{}^g)^g = \tilde{z}_2{}^g = \tilde{z}_1$ and $((\tilde{z}_2)^g)^g = (\tilde{z}_1)^g = \tilde{z}_2$ $\quad g^2 \varepsilon C(\tilde{z}_1)$ and $g^2 \varepsilon C(\hat{z}_2)$ $g^2 \varepsilon C(<\tilde{z}_1,\tilde{z}_2>)$ 2(a) is proved. We now prove 2(c) first: for each $i=1,2,3,\ldots,r=p^{d-1}$, let $g_i$ be defined on the set $\{[vz^i,a,b]|\ a \varepsilon \Omega, b \varepsilon \Omega\} = \{vz^i\} x\ \Omega\ x \Omega$ by

(10.7) $\quad g_i:[vz^i,a,b] \to [vz^i,b,a]$ all a,b. While leaves all vectors outside $\{vz^i\}\Omega x\ \Omega\ x\ \Omega$ fixed; then obviously $g=g_1 g_2 \ldots g_r$ and since $t_{vz^i}:[vz^i,a,b] \to (vz^i,a+1,b]$ for all a,b $(t_{vz^i})^{g_i}:[vz^i,a,b]g_i \to [vz^i,a+1,b]g_i$ ; i.e. $(t_{vz^i})^{g_i}:[vz^i,b,a,] \to [vz^i,b,a+1]$ all a,b by the

definition if $g_i$, but this is the same as the action of $t_{vz^i*}$ (see definition 6.4). Hence,

(10.8) $\quad (t_{vz^i})^{g_i} = t_{vz^i*}$

likewise $(t_{vz^i*})^{g_i} = t_{vz^i}$ hence

$g_i \varepsilon N(<t_{vz^i}, t_{vz^i*}>)$ 2(c) is proved.

For part (b) because

$\prod_{i=1}^{r} t_{vz^i}:[vz^i,a,b] \to [vz^i,a+1,b]$ all i; a,b.

$(\prod_{i=1}^{r} t_{vz^i})^g:[vz^i,a,b]g \to [vz^i,a+1,b]g$ all i; a,b.

$(\prod_{i=1}^{r} t_{vz^i})^g:[vz^i,b,a] \to [vz^i,b,a+1]$ all i; a,b.

this is the same as the action of $\prod_{i=1}^{r} t_{vz^i*}$

(10.9) $\quad$ So, $(\prod_{i=1}^{r} t_{vz^i})^g = \prod_{i=1}^{r} t_{vz^i*}$

similarly $(\prod_{i=1}^{r} t_{vz^i*})^g = \prod_{i=1}^{r} t_{vz^i}$ ;hence, (b) is proved.

Now to prove the theorem, given two conjugate elements x and y in $S_{pn}$, we will find two elements x* and $y^* \varepsilon S_{pn}$ such that (i) x is conjugate to x*,(ii) $\bar{x}$ is conjugate to $\bar{x}^*$, (iii) y is conjugate to y*,(iv) $\bar{y}$ is conjugate to $\bar{y}^*$ and (v) x*,y* depend only upon the number of cycles of x and y of all lengths and types; that is, x* depends only on the numbers $x^d, x_1^d, x_2^d$ for all d=0,1,2,...,n, and y* depends only on $y^d, y_1^d, y_2^d$ for all d=0,1,2,...,n, (see proof of lemma 3.2 for the notations). And the most important of all, (vi) x* and y* are locally conjugate in the families $F_1$ and $F_2$ defined in (9.1),(9.2).

(10.10) $\quad$ For each i=1,2,...,n, the number of all vectors in cycles of x of lengths less than $p^i$ is a multiple of $p^i$,

let it be $\delta_i . p^i$. <u>We first assume that for every i, $1 \leq \delta_i < p$.</u> Let $A_i = \{1, 2, \ldots, \delta_i\}$ and let $E_1 = \{1\}$ . $E_j = E_1 x E_1 x, \ldots E_1$, j times. The assumption $1 \leq \delta_i < p$ for all i implies $\delta_n = 1$ and x,y are products of cycles of lengths less than or equal $p^{n-1}$ and x,y own $p^{i-1}$-cycles for i=1,2,...,n. So obviously x,y are not $p^n$-cycles. However, if both x,y are $p^n$-cycles, then they are both type II $p^n$-cycles. Hence by theorem 1 they are conjugate in N(T). For general case beyond the condition $1 \leq \delta_i < p$ all i, the proof is similar to what we are going to do below, hence, is omitted.

Define an increasing sequence of subsets of $\Omega_n$ as follows:

$$\Delta_1 = E \times E \times \ldots \quad . \times E \times E \times A_1 \times \Omega = E_{n-2} \times A_1 \times \Omega_1$$

$$\Delta_2 = E \times E \times \ldots \quad . \times E \times A_2 \times \Omega \quad \times \Omega = E_{n-3} \times A_2 \times \Omega_2$$

(10.11)

$$\Delta_d = E \times E \times \ldots . \times E \times A_d \times \Omega \times \Omega \times \ldots . \times \Omega = E_{n-d-1} \times A_d \times \Omega_d$$

$$\Delta_{n-1} = A_{n-1} \times \Omega \times \Omega \times \ldots \quad . \times \Omega = A_{n-1} \times \Omega_{n-1}$$

$$\Delta_n = \Omega_n$$

We will define the permutation x*,y* using $\Delta_i$ as vectors in cycles of x* and y* of lengths less than $p^i$ in such a way that the corresponding cycles of x and x* are of the same type and so are the corresponding cycles of y and y*.

For each d=1,2,...,n-2. If a $p^d$-cycle of $\bar{x}$ induces

a type I $p^d$-cycle of $x$, then it induces exactly $p$ type I $p^d$-cycles of $x$ (see lemma 3.1). That is, $x_1^d$ is divisible by $p$. Let $x_1^d = p.m_d^x$ (this $m_d^x$ is the same as $\bar{m}_1^d$ in (9.10) and (9.11)). Now the total number of vectors in $p^d$-cycles of $x$ is

(10.12) $\quad p^d \,.\, X^d = \delta_{d+1}p^{d+1} - \delta_d.p^d$ (by the definition (10.10) of $\delta_d$.)

$$= (p-\delta_d).p^d + (\delta_{d+1}-1).p^{d+1}$$
$$= p^d.x_1^d + p^d.x_2^d$$
$$= p^d.(p.m_d^x) + p^d.x_2^d$$
$$= p^{d+1}.m_d^x + p^d.x_2^d \qquad \text{hence,}$$

(10.13) $\quad p^d x_2^d - (p-\delta_d).p^d = ((\delta_{d+1}-1)-m_a^x).p^{d+1} \equiv 0$ (modulo $p^{d+1}$) likewise, for $y$ we have,

(10.14) $\quad p^d y_2^d - (p-\delta_d).p^d = ((\delta_{a+1}-1)-m_d^y).p^{d+1} \equiv 0$ (modulo $p^{d+1}$)

(10.15) $\quad$ hence, $x_2^d \equiv y_2^d \equiv p-\delta_d$ (mod p). Let

(10.16) $\quad \Phi_d = \underbrace{ExEx\ldots xE}_{(n-d-1)\text{ times}}x(\Omega-A_d)x\underbrace{\Omega x\Omega x\ldots x\Omega}_{d\text{ times}}$ , then

(10.17) $\quad |\Phi_d| = (p-\delta_d).p^d$

then the number of vectors contained in type IIp$^d$-cycles of $x$ less $|\Phi_a|$ will be a multiple of $p^{d+1}$ (by (10.13), and the same for $y$ by (10.14). Define

(10.18) $\quad \theta_d = \underbrace{ExEx\ldots xE}_{(n-d-2)\text{ times}}x(A_{d+1}-E)x\underbrace{\Omega x\Omega x\ \ldots x\Omega}_{(d+1)\text{ times}}$ then

(10.19) $\quad |\theta_d| = (\delta_{d+1}-1).p^{d+1}$

Note that both sets $\Phi_d$ and $\theta_d$ work simultaneously for both $x$ and $y$ because they depend only on $\delta_d$'s and $A_d$'s which are the same for both $x$ and $y$. It is easy to see:

(i) $\Phi_d$ and $\theta_d$ are disjoint

(ii) $\Delta_{a+1} - \Delta_d = \Phi_d \cup \theta_d$

(iii) If we define $\Phi_{n-1} = (\Omega - A_{n-1}) \times \Omega_{n-1} = \Delta_n - \Delta_{n-1}$ then $\Omega_n = \Delta_n = \bigcup_{d=1}^{n-1} \Phi_d \bigcup_{d=1}^{n-2} \theta_d$ disjoint union.

We will use the vectors in $\Phi_d$ as vectors in some $(p-\delta_d)$ typeII $p^d$-cycles of x* and y* and use $(\delta_{a+1}-1-m_d^x).p^d$ vectors of $\theta_d$ as other $(\delta_{d+1}-1-m_d^x)$ type II $p^d$-cycles of x*, then the last $m_d^x.p^d$ vectors of $\theta_d$ as $m_d^x$ type I $p^d$-cycles of x*. Likewise, $(\delta_{d+1}-1-m_d^y)p^d$ vectors of $\theta_d$ will be used to form type II $p^d$-cycles of y* other than the $(p-\delta_d)$ mentioned above, and the remaining $m_d^y.p^d$ vectors of $\theta_d$ will be used to form all type I $p^d$-cycles of y*. See figure 4.3

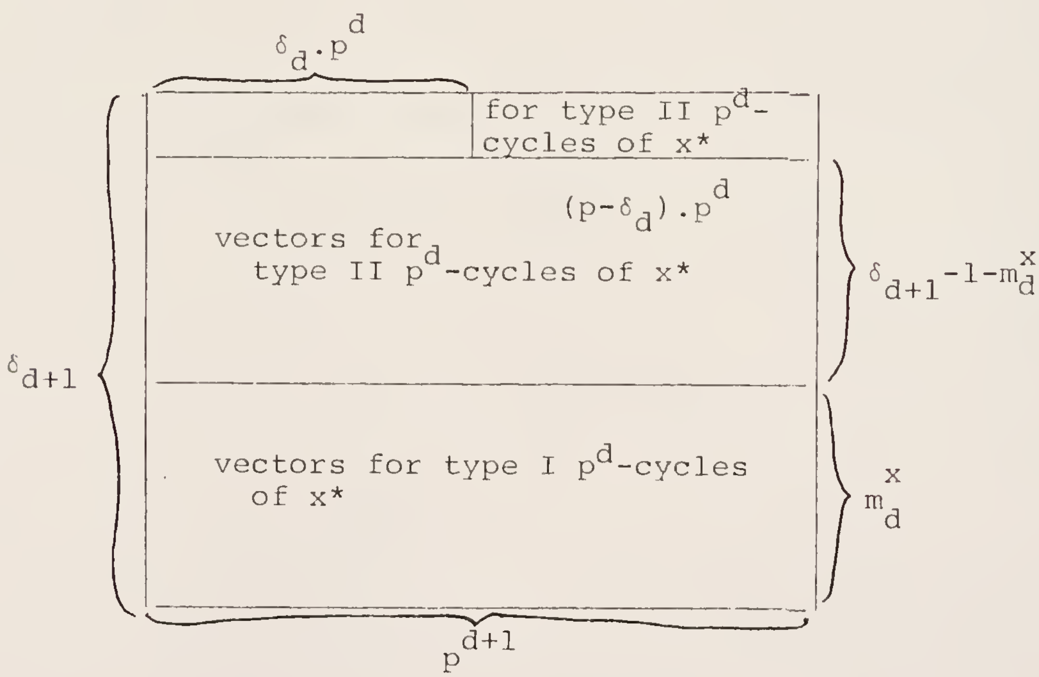


Figure 4.3

Now write x,y as products of cycles in an order that cycles of smaller lengths first and in cycles of the same

length, type II cycles first, namely the order is: fixed points, type II cycles, type I p-cycles, type II $p^2$-cycles, type I $p^2$-cycles, and so on. This way we get a one-to-one correspondence between cycles of x and y preserving the cycle length. Although this correspondence does not preserve the type but for each d, the first $(p-\delta_d)\cdot p^d$-cycles of both x and y have been ordered that they are all type II. By (10.15) the number of $p^d$-cycles which x,y are of different type is a multiple of p namely, $(m_d^x - m_d^y).p$.

For each $d=1,2,\ldots,n-1$, define $x_d^*$ and $y_d^*$ on $\Phi_d \cup \Theta_d = \Delta_{d+1}-\Delta_d$ (let $\Theta_{n-1} = \phi$) inductively as follows:

$x_1^*, y_1^*$ fix all vectors in $\Delta_1$. On $\Phi_1 = E_{n-2}x(\Omega-A)x\Omega_1$ both $x_1^*$ and $y_1^*$ map $E_{n-2}x[a,b]$ to $E_{n-2}x[a,b+1]$ for all $a\varepsilon\Omega-A$, $b\varepsilon\Omega$. Thus, $x_1^*$, $y_1^*$ act on $\Phi_1$ as products of $(p-\delta_1)$ type II p-cycles. Let $B_2^x = \{2,3,\ldots,\delta_2-m_1^x\}$ and $C_2^x = A_2-B_2^x-\{1\}$;then $|B_2^x| = \delta_2-1-m_1^x$, $|C_2^x| = m_1^x$ and define $B_2^y, C_2^y$ similarly. Then, let $x_1^*$ map $E_{n-3}x[a,b,c]$ to $E_{n-3}x[a,b,c+1]$ for all $b,c\varepsilon\Omega$ and all $a\varepsilon B_2^x$ and map $E_{n-3}x[a,b,c]$ to $E_{n-3}x[a,b+1,c]$ for all $b,c\varepsilon\Omega$ and all $a\varepsilon C_2^x$. Thus, $x_1^*$ is defined on $\Delta_2$ so that the number of type I (type II) p-cycles is the same as that of x. The definition of $y_1^*$ on $\Delta_2$ is similar. Obviously $x_1^*$ and $y_1^*$ are conjugate.

Inductively, assume $x_{d-1}^*$ and $y_{d-1}^*$ are defined on $\Phi_{d-1} \cup \Theta_{d-1} = \Delta_d-\Delta_{d-1}$;both are products of $p^{d+1}$-cycles and are conjugate and $x_{d-1}^*$ contains as many type I $p^{d+1}$-cycles as x does; $y_{d-1}^*$ contains as many type I

$p^{d+1}$-cycles as y does (same for type II $p^{d-1}$cycles).

We then define $x_d^*$ and $y_d^*$ on $\Phi_d \cup \Theta_d = \Delta_{d+1} - \Delta_d$ as follows: On $\Phi_d = E_{n-d-1} \times (\Omega - A_d) \times \Omega_d$, let $z_d$ be the permutation on $\Omega_d$ of order $p^d$ (see lemma 4.1). Define $x_d^*$ and $y_d^*$ the same on $\Phi_d$ by sending vector $E_{n-d-1} \times [a, w^d]$ to $E_{n-d-1} \times [a, w^d z_d]$ for all $a \varepsilon \Omega - Ad$; then it is easily seen that both $x_d^*$ and $y_d^*$ act the same as products of $(p - \delta_d)$ type II $p^d$-cycles.

For each $c \varepsilon A_{d+1} - \{1\}$ let $w = E_{n-d-2} \times \{c\} \varepsilon \Omega_{n-d-1}$ and let $\Delta = \{w\} \times \Omega_{d-1} \varepsilon \Omega_{n-2}$ which are defined the same as in lemma 4.1. Then by this lemma, there exists $z \varepsilon S_p^{n-2}$ which is a $p^{d-1}$-cycle on $\Delta$ (note that $|\Delta| = p^{d-1}$), and choose a vector $v \varepsilon \Delta$, let $r = p^{d-1}$ as in lemma 4.2. Note also that the vector v, the element z and the set $\Delta$ all depend on the number $c \varepsilon A_{d+1} - \{1\}$.

(i) If $c \varepsilon B_d^x \cap B_d^y$. We defined both $x^*$ and $y^*$ on $E_{n-d-2} \times c \times \Omega_{d+1} = \Delta \times \Omega \times \Omega$ as $\tilde{z}_2$ defined in (10.2). Hence, both are products of p type II $p^d$-cycles.

(ii) If $c \varepsilon B_d^x - B_d^y$ then define $x^*$ to be $\tilde{z}_2$ as in (10.2) and define $y^*$ to be $\tilde{z}_1$, in (10.1) on the set $\Delta \times \Omega \times \Omega$.

(iii) If $c \varepsilon B_d^y - B_d^x$ then define $x^*$ to be $\tilde{z}_1$, in (10.1) and $y^*$ to be $\tilde{z}_2$ in (10.2).

(iv) If $c \varepsilon A_{d+1} - 1 - B_d^x - B_d^y$ define both $x^*$ and $y^*$ as $\tilde{z}_1$ (in (10.1).

Since different c gives different $\tilde{z}_i$'s i=1,2,... on different (also disjoint) sets $\Delta$'s. As c ranges over the set $A_{d+1} - \{1\}$, the union of all $\Delta$'s equals $\Theta d$ (which is defined in (10.18)). We have:

(a) $x_d^*$ contains as many type I/type II $p^d$-cycles as $x$ does. $y_d^*$ contains as many type I/type II $p^d$-cycles as $y$ does.

(b) For each $c \varepsilon A_{d+1} - \{1\}$, on the set $E_{n-d-2} \times \{c\} \times \Omega_{d+1} = \Delta \times \Omega \times \Omega$, either $x_d^*$ and $y_d^*$ are the same (case (i)(iv) above) or either one of $x_d^*$ or $y_d^*$ acts as $\tilde{z}_1$ while the other one acts as $\tilde{z}_2$ defined in (10.1) and (10.2). For c of the latter case let $g_d^c$ be defined as in lemma 4.2 part (2) on the set $E_{n-d-2} \times \{c\} \times \Omega_{d+1} = \Delta \times \Omega \times \Omega$. Then by part 2(b)(c) $g_d^c \varepsilon N(< \prod_{i=1}^{r} t_{vz^i}, \prod_{i=1}^{r} t_{vz^{i*}} >)$ or $g_d^c$ factors into product of r elements each lies in $N(<t_{vz^i}, t_{vz^{i*}}>)$ for some i. Let $g_d^c$ be identity map if c belongs to $B_d^x \cap B_d^y$ or $A_{d+1} - \{1\} - B_d^x - B_d^y = C_d^x \cap C_d^y$. Hence,

$$(x_d^*)^{\prod_{c \varepsilon A d+1 - \{1\}} g_d^c} = y_d^*$$

By induction $x_d^*$ and $y_d^*$ are defined; hence, let $x^* = \prod_{d=1}^{n-1} x_d^*$, $y^* = \prod_{d=1}^{n-1} y_d^*$; then x is conjugate to $x^*$ and y is conjugate to $y^*$ also by proposition 3.4. $\bar{x}$ and $\bar{x}^*$ are conjugate, $\bar{y}$ and $\bar{y}^*$ are conjugate. Hence, conjugating elements $g_1, g_2, g_3$ can be found such that $x^{g_1} = x^*$ with $g_1 \varepsilon N(T)$, $(y^*)^{g_3} = y$ with $g_3 \varepsilon N(T)$ and $(x^*)^{g_2} = Y^*$ where $g_2$ factors into product of elements each either comes from members of $F_1$ or members of $F_2$. In fact, $g_2 = \prod_d \prod_c g_d^c$. The proof of theorem 2 is then completed.

## 11. Fusion in Direct Products

Since the Sylow p-subgroups of the symmetric groups of degree m, where m is any positive integer, are the direct

products of the groups $S_{p^n}$'s for various n's as discussed in the preceding sections, to know the fusion of elements in such sylow subgroups, it suffices to know the fusion of elements in the direct product of two $S_{p^n}$'s. In this section we will discuss the local fusion of the direct product of two sylow subgroups $S_{p^n}$'s with the same degree $p^n$. For discussion of direct product of two $S_{p^n}$'s with different n's, the proof is similar, hence, is omitted.

Before going into direct product, let's look at a single sylow group $S_{p^n}$ for a moment. Recall each conjugacy class of $S_{p^n}$, except the class containing long $p^n$-cycles, corresponds to a cycle decomposition, namely, a n-tuple of constants (non-negative integers) $\{C_0, C_1, C_2, \ldots, C_{n-1}\}$ satisfying the relation

$$C_0 + pC_1 + p^2C_2 + \ldots + p^{n-1}C_{n-1} = p^n$$

where for each $d=0,1,2,\ldots,n-1$. $C_d$ is the number of $p^d$-cycles in the cycle decomposition. Note that $C_0+pC_1+p^2C_2+\ldots+p^dC_d = \delta_{d+1}.p^{d+1}$ as defined in (10.10) which is the number of vectors in cycles of lengths less than $p^{d+1}$. Again, we assume $1 \leq \delta_i < p$ for all $i=1,2,\ldots,n$ and construct the increasing sequence of sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ as in (10.11). Then we define the permutation $x^*$ using $\Delta_i$ as set of vectors in cycles of $x^*$ of lengths less than $p^i$ as we did in section 10 with the choice of the element x in the conjugacy class such that all cycles of x are type II (except $p^o$-cycles). Thus all cycles of $x^*$ are type II except $p^o$-cycles and $x^*$ is uniquely determined by the

conjugacy class. We call this unique element $x^*$ of the conjugacy class the <u>standard element</u> of this conjugacy class. The example below shows how the standard element of a conjugacy class is found:

EXAMPLE 4.3: Let $p=3$, $n=4$ the conjugacy class corresponds to the constants $C_0=12, C_1=2$, $C_2=4$, $C_3=1$, contains the standard element $x^*$ =

([1111])([1112])([1113])([1121])([1122])([1123])([1131])([1132])([1133])
([1211])([1212])([1213])([1221] [1222] [1223])([1231] [1232] [1233])
([1311] [1321] [1331] [1312] [1322] [1332] [1313] [1323] [1333])
([2111] [2121] [2131] [2112] [2122] [2132] [2113] [2123] [2133])
([2211] [2221] [2231] [2212] [2222] [2232] [2213] [2223] [2233])
([2311] [2321] [2331] [2312] [2322] [2332] [2313] [2323] [2333])
([3111] [3211] [3311] [3121] [3221] [3321] [3131] [3231] [3331]
[3112] [3212] [3312] [3122] [3222] [3322] [3132] [3232] [3332]
[3113] [3213] [3313] [3123] [3223] [3323] [3133] [3233] [3333])

Notice that this example goes beyond the condition $1 \le \delta_i < p$. We can easily see that this condition is not substantial and hence the standard element can be found in general.

Now let $S$ and $S'$ be Sylow p-subgroups of the symmetric groups $\gamma_{p^n}$ and $\gamma_{p^n}'$ acting on the set $\Omega_n$ and $\Omega_n'$ respectively. Consider the group $\mathcal{S} = S \times S'$, it is easy to see $\mathcal{S}$ is a Sylow p-subgroup of $\gamma_{2.p^n}$ acting on the set $\Omega_n \dot{\cup} \Omega_n'$ (disjoint union), and it is also the Sylow p-subgroup of the subgroup $\gamma_{p^n} \times \gamma_{p^n}'$ of $\gamma_{2.p^n}$ where $\gamma_{p^n} \times \gamma_{p^n}'$ is considered as the subgroup which "stabilizes" both sets $\Omega_n$ and $\Omega_n'$; that is, the permutations which maps elements of $\Omega_n$ into $\Omega_n$, elements of $\Omega_n'$ into $\Omega_n'$.

Let the notations (5.6)-(5.10), (6.1)-(6.7), (9.1) and (9.2) be defined on S as well as on S' with the lash "$'$" added to all those in S'. Consequently, we have notations for "products", namely, we have $A^i \times A^{i'}$, $i=0,1,\ldots n-1$, in particular $T \times T'$ = the group generated by all $t_v(n-1)$'s and $t_v(n-1)'$ s. Also, besides $F_1, F_2$, $F_1', F_2'$, we may define

(11.1) $\quad F_1^* = \{N(T\times T')\} \cup F_1 \cup F_1'$

(11.2) $\quad F_2^* = \{N(T\times T')\} \cup F_2 \cup F_2'$

Now let $xx'$ and $yy'$ be two conjugate elements of $\mathcal{S} = S \times S'$ where $x,y \in S$ and $x',y' \in S'$. Let $x^*$, $x'^*$, $y^*$, $y'^*$ be the <u>standard elements</u> of the conjugacy classes containing $x,x',y,y'$ respectively; then, by the main theorem, there exist $g_1, g_1'$, $g_2, g_2'$ which factor into products of elements in the family $F_1, F_1', F_1, F_1'$ respectively (or $F_2, F_2', F_2, F_2'$ respectively) such that

$$x^{g_1} = x^* , \quad (x')^{g_1'} = x'^* ,$$
$$y^{g_2} = y^* , \quad (y')^{g_2'} = y'^*$$

hence

(11.3) $$(xx')^{g_1 g_1'} = ((xx')^{g_1})^{g_1'} = (x^{g_1}.x'^{g_1})^{g_1'}$$
$$= (x^*.x')^{g_1'} = (x^*)^{g_1'} . x'^{g_1'} = x^*.x'^*$$

Here the equalities hold because $g_1$ centralizes $x'$ and $g_1'$ centralizes $x^*$, likewise

(11.4) $\quad (yy')^{g_2 g_2'} = y^* y'^*$

Thus $F_1^*$ (and $F_2^*$, too) control the local fusion from $xx'$ to $x^*x'^*$ and from $yy'$ to $y^*y'^*$. Hence, the last problem is how the element $y^*y'^*$ and $x^*x'^*$ are <u>fused</u> together. We claim in the following lemma that the normalizer $(N(T\times T'))$

will give the local fusion from $x^*x'^*$ to $y^*y'^*$ (that is the reason why we throw $N(T\times T')$ into both definitions of $F_1^*$ and $F_2^*$):

LEMMA 4.4 <u>Let $x,y\in S$, $x',y'\in S'$, if $xx',yy'\in \mathcal{S}= S\times S'$ are conjugate and if $x^*,x'^*,y^*,y'^*$ are standard elements of the conjugacy classes containing $x,x',y,y'$ respectively, then there is $g\in N(T\times T')$ such that</u>

<u>$(x^*x'^*)^g=y^*y'^*$</u>.

PROOF Since we will be dealing with the four elements $x^*,x'^*,y^*,y'^*$ through the proof, to avoid writing the "*" all the time, we will use $x,x',y,y'$ to denote $x^*,x'^*$ $y^*,y'^*$. That is, we assume $x,x',y,y'$ are <u>standard elements</u> themselves such that $xx'$ is conjugate to $yy'$.

Let $\{C_o^x, C_1^x, \ldots, C_{n-1}^x\}$, $\{C_o^y, C_1^y, \ldots, C_{n-1}^y\}$ $\{C_o^{x'}, C_1^{x'}, \ldots, C_{n-1}^{x'}\}$, $\{C_o^{y'}, C_1^{y'}, \ldots, C_{n-1}^{y'}\}$ be the constants corresponding to these conjugate classes. We have, because $xx'$ is conjugate to $yy'$, that for each $i=0,1,2,\ldots,n-1$

(11.5) $\quad -(C_i^x-C_i^y)=C_i^{x'}-C_i^{y'}$

For $i=0$, we have $C_o^y-C_o^x=C_o^{x'}-C_o^{y'}$; let $V_o$ be the set of vectors in $\Omega$ which are fixed by exactly one of the two elements $xx'$ and $yy'$, and let $V_o'$ be the set of vectors in $\Omega'$ which are fixed by exactly one of the two elements $xx'$ and $yy'$. By the way the standard elements are defined, it is not too difficult to see that $|V_o| = |V_o'| = |C_o^y-C_o^x| = |C_o^{x'}-C_o^{y'}| = f_o$. We call vectors in $V_o$ and $V_o'$ "extra" vectors of $p^o$-cycles of $xx'$ and $yy'$.

They form "blocks" $\{v\times\Omega\}$ and $\{w'\times\Omega'\}$ for some v and w' in $\Omega_{n-1}$ (recalling all $C_0^{x}, C_0^{x'}, C_0^{y}, C_0^{y'}$ are multiples of p. Hence, $f_o$ is also a multiple of p).

Now let $g_1$ be the permutation, which acts on $V_0 \cup V_0'$ and leaves all other vectors fixed, defined by sending blocks onto blocks, and in each pair of corresponding blocks $v\times\Omega$ and $w'\times\Omega'$ it maps $[v,a] \to [w',a]$ all $a\in\Omega$. Then, $(xx')^{g_1}$ and yy' have the same set of fixed vectors and obviously $g \in N(T\times T')$ because if g interchanges the blocks $v\times\Omega$ and $w'\times\Omega'$ then $(t_v)^g = t_w'$ and $(t_w')^g = t_v$. We have used $g_1$ as conjugating element to xx' to pass from xx' to the element $(xx')^{g_1}$ which is also a product of standard elements and has the same fixed vectors as fixed vectors of yy' in the sense that they are closer "looks."

Inductively, we find elements $g_1, g_2, \ldots, g_d$ all in $N(T\times T')$ such that $(xx')^{g_1 g_2, \ldots g_d}$ and yy' have the same vectors in all cycles of lengths less than $p^d$; then let $V_d$ be the set of vectors in $\Omega$ which lie in one and only one $p^d$-cycle of either $(xx')^{g_1, \ldots g_d}$ or yy', and let $V_d'$ be defined likewise. Then, define $g_{d+1}$ to be the permutation acting on $V_d \cup V_d'$ leaving all other vectors fixed by interchanging vectors in $p^d$-cycles of $(xx')^{g_1, \ldots g_d}$ and yy', thus making the elements $(xx')^{g_1, \ldots g_{d+1}}$ and yy' have the same vectors in all cycles of lengths less than $p^{d+1}$. By induction, we have the elements $g_1, g_2, \ldots g_n$ all in $N(T\times T')$. Hence, $g = g_1 g_2 \ldots g_n \in N(T\times T')$ such that $(xx')^g = yy'$. Conclude the proof of the lemma.

Let's look at an example which illustrates the proof of the lemma;

EXAMPLE 4.5 Let p=3, n=3, in $S_{27} \times S_{27}$. The standard elements x, x', y, y' are given by the constants:

| d | $c_d^x$ | $c_d^{x'}$ | $c_d^y$ | $c_d^{y'}$ |
|---|---|---|---|---|
| 0 | 3 | 9 | 6 | 6 |
| 1 | 2 | 3 | 4 | 1 |
| 2 | 2 | 1 | 1 | 2 |

We first write the cycle decomposition of xx' and yy' then underline the cycles of xx' which are contained in yy', and find the sets $V_0$ and $V_0'$ hence the conjugating element $g_1$. Secondly, write the cycles of $(xx')^{g_1}$ and underline the cycles of $(xx')^{g_1}$, which are contained in yy'. Hence, find the sets $V_1$ and $V_1'$ and then the conjugating element $g_2$. In this example, the sets $V_2$ and $V_2'$ are empty;hence, $(xx')^{g_1 g_2} = yy'$. The local conjugation takes two steps to go from xx' to yy'. See the following page.

xx' = ([111])([112])([113])([111]')([112]')([113]')([121]')([122]')([123]')([131]')([132]')([133]')([121] [122] [123])([131] [132] [133])([211]' [212]' [213]')([221]' [222]' [223]')([231]' [232]' [233]')
([211] [221] [231] [212] [222] [232] [213] [223] [233])([311] [321] [331] [312] [322] [332] [313] [323] [333])([311] [321] [331] [312] [322] [332] [313] [323] [333])

yy' = ([111])([112])([113])([121])([122])([123])([111]')([112]')([113]')([121]')([122]')([123]')([131] [132] [133])([211] [212] [213])([221] [222] [223])([231] [232] [233])([131]' [132]' [133]')
([311] [321] [331] [312] [322] [332] [313] [323] [333])([211]' [221]' [231]' [212]' [222]' [232]' [213]' [223]' [233]')([311]' [321]' [331]' [312]' [322]' [332]' [313]' [323]' [333]')

$V_o$ = { [121] [122] [123] }
$g_1 \updownarrow$ ↕ ↕ ↕
$V'_o$ = { [131] [132] [133] }

$(xx')^{g_1}$ = ([111])([112])([113])([111]')([112]')([113]')([121]')([122]')([123]')([121])([122])([123])([131]' [132]' [133]')([131] [132] [133])([211]' [212]' [213]')([221]' [222]' [223]')([231]' [232]' [233]')
([211] [221] [231] [212] [222] [232] [213] [223] [233])([311] [321] [331] [312] [322] [332] [313] [323] [333])([311]' [321]' [331]' [312]' [322]' [332]' [313]' [323]' [333]')

$V_1$ = { [211] [212] [213] [221] [222] [223] [231] [232] [233] }
$g_2 \updownarrow$ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕
$V'_1$ = { [211] [212] [213] [221] [222] [223] [231] [232] [233] }

$(xx')^{g_1 g_2}$ = ([111])([112])([113])([111]')([112]')([113]')([121]')([122]')([123]')([121])([122])([123])([131]' [132]' [133]')([131] [132] [133])([211] [212] [213])([221] [222] [223])([231] [232] [233])
([211]' [221]' [231]' [212]' [222]' [232]' [213]' [223]' [233]')([311] [321] [331] [312] [322] [332] [313] [323] [333])([311]' [321]' [331]' [312]' [322]' [332]' [313]' [323]' [333]')

As a corollary of the main theorem and this lemma, we have:

CONCLUSION: In the Sylow p-subgroups of a symmetric group of any degree, the local fusion of elements is given by the normalizers of groups of only two different structures; one is a big elementary abelian group of p-power rank, the other is the small elementary abelian group of order $p^2$.

APPENDIX

In this appendix we will see four examples of permutations x, x*, y, y* in $S_p n$ with p=5, n=3 and x, x*, y, y* are all conjugate in $\gamma_p n$ such that $\bar{x}$ is conjugate to $\bar{x}^*$. These examples illustrate: (1) how the conjugating element $g_1 \varepsilon N(T)$ is found to make $x^{g_1}=x^*$; (2) how the element $g_{2'} \varepsilon N(T)$ is found to make $(y^*)^{g_3}=y$; (3) how the element $g_2$ is found so that $(x^*)^{g_2}=y^*$ and $g_2$ factors into product of elements in the normalizers of the family of p-subgroups of the form $<t_{v(n-2)}, t_{v(n-2)}>$, with $v^{(n-2)}$ ranges overthe set $\Omega_{n-2}$ mentioned in the main theorem.

In fact, how the elements x* and y* can be found from x,y as mentioned in Theorem 2 can also be seen implicitly from these examples.

Each example gives the element first by the maps $x_0, x_1, x_2, \ldots x_{n-1}$ as defined in (5.9), then by its cycle decomposition. All cycle decompositions are written in the same manner such that the corresponding cycles all have the same length and/or the same type. The type of each cycle (or cycles) is put on the top of the cycle (or cycles). The elements $g_i$ are defined to map the vectors into the vectors in the corresponding positions of the other element. The illustration of how $g_1 \varepsilon N(T)$ and how $g_2$ factors

into product of elements in $N(\langle t_{v(n-2)}, t^*_{v(n-2)}\rangle)$ can be seen to some extent.

Example 1. p=5, n=3, $x= \{x_o, x_1, x_2\}$

$x_o=0$, $\quad v_1 = \boxed{1\,|\,2\,|\,3\,|\,4\,|\,5}$, $\quad x_1(v_1^1) = \boxed{2\,|\,0\,|\,2\,|\,0\,|\,0}$ $\qquad \dfrac{[v_1, v_2]}{x_2[v_1, v_2]} =$

| | | | | |
|---|---|---|---|---|
| [1,1] | [1,2] | [1,3] | [1,4] | [1,5] |
| 2 | 3 | 1 | 4 | 5 |
| [2.1] | [2,2] | [2,3] | [2,4] | [2,5] |
| 1 | 2 | 1 | 0 | 1 |
| [3,1] | [3,2] | [3,3] | [3,4] | [3,5] |
| 2 | 4 | 3 | 2 | 1 |
| [4,1] | [4,2] | [4,3] | [4,4] | [4,5] |
| 2 | 3 | 2 | 1 | 2 |
| [5,1] | [5,2] | [5,3] | [5,4] | [5,5] |
| 2 | 0 | 3 | 1 | 4 |

Cycle Structures:

$\bar{x}$ = ..... $\quad$ x = ..... ..... ____ ____ ____

.....

.....

([241])([242])([243])([244])([245])([521])([522])([523])([524])([525])([211] [212] $\overset{\text{II}}{[213]}$ [214] [215])([221] [223] $\overset{\text{II}}{[225]}$ [222] [224])([231] [232] [233] [234] [235])

([251] [252] $\overset{\text{II}}{[253]}$ [254] [255])([511] [513] $\overset{\text{II}}{[515]}$ [512] [514])([531] [534] $\overset{\text{II}}{[532]}$ [535] [533])([541] [542] $\overset{\text{II}}{[543]}$ [544] [545])([551] [555] [554] [553] [552])

([411] [412] $\overset{\text{II}}{[413]}$ [414] [415])([421] [424] $\overset{\text{II}}{[422]}$ [425] [423])([431] [433] $\overset{\text{II}}{[435]}$ [432] [434])([441] [442] $\overset{\text{II}}{[443]}$ [444] [445])([451] [453] [455] [452] [454])

([111] [133] $\overset{\text{I}}{[154]}$ [124] [142])([112] [134] $\overset{\text{I}}{[155]}$ [125] [143])([113] [135] $\overset{\text{I}}{[151]}$ [121] [144])([114] [131] $\overset{\text{I}}{[152]}$ [122] [145])([115] [132] [153] [123] [141])

([311] [333] [351] [322] [341] [313] [335] [353] [324] [343] [315] [332] $\overset{\text{II}}{[355]}$ [321] [345] [312] [334] [352] [323] [342] [314] [331] [354] [325] [344])

Example 2. p=5, n=3, $x^* = \{x_o^*, x_1^*, x_2^*\}$

$x_o=0$, $\quad v_1 = \boxed{1\,|\,2\,|\,3\,|\,4\,|\,5}$, $\quad x_1^*(v_1^1) = \boxed{0\,|\,0\,|\,0\,|\,1\,|\,1}$ $\qquad \dfrac{[v_1, v_2]}{x_2^*[v_1, v_2]} =$

| | | | | |
|---|---|---|---|---|
| [1,1] | [1,2] | [1,3] | [1,4] | [1,5] |
| 0 | 0 | 1 | 1 | 1 |
| [2,1] | [2,2] | [2,3] | [2,4] | [2,5] |
| 1 | 1 | 1 | 1 | 1 |
| [3,1] | [3,2] | [3,3] | [3,4] | [3,5] |
| 1 | 1 | 1 | 1 | 1 |
| [4,1] | [1,2] | [4,3] | [4,4] | [4,5] |
| 0 | 0 | 0 | 0 | 0 |
| [5,1] | [5,2] | [5,3] | [5,4] | [5,5] |
| 0 | 0 | 0 | 0 | 1 |

Cycle Structures:

$\bar{x}^*$ = ..... $\quad$ $x^*$ = ..... ..... ____ ____ ____

.....

.....

([111])([112])([113])([114])([115])([121])([122])([123])([124])([125])([131] [132] $\overset{\text{II}}{[133]}$ [134] [135])([141] [142] $\overset{\text{II}}{[143]}$ [144] [145])([151] [152] [153] [154] [155])

([211] [212] $\overset{\text{II}}{[213]}$ [214] [215])([221] [222] $\overset{\text{II}}{[223]}$ [224] [225])([231] [232] $\overset{\text{II}}{[233]}$ [234] [235])([241] [242] $\overset{\text{II}}{[243]}$ [244] [245])([251] [252] [253] [254] [255])

([311] [312] $\overset{\text{II}}{[313]}$ [314] [315])([321] [322] $\overset{\text{II}}{[323]}$ [324] [325])([331] [332] $\overset{\text{II}}{[333]}$ [334] [335])([341] [342] $\overset{\text{II}}{[343]}$ [344] [345])([351] [352] [353] [354] [355])

([411] [421] $\overset{\text{I}}{[431]}$ [441] [451])([412] [422] $\overset{\text{I}}{[432]}$ [442] [452])([413] [423] $\overset{\text{I}}{[433]}$ [443] [453])([414] [424] $\overset{\text{I}}{[434]}$ [444] [454])([415] [421] [435] [445] [455])

([511] [521] [531] [541] [551] [512] [522] [532] [542] [552] [513] [523] $\overset{\text{II}}{[533]}$ [543] [553] [514] [524] [534] [544] [554] [515] [525] [535] [545] [555])

Example 3. $p=5$, $n=3$, $y^*=\{y_o^*, y_1^*, y_2^*\}$

$y_o^* = 0$, $\begin{array}{r|c|c|c|c|c|} v_1 = & 1 & 2 & 3 & 4 & 5 \\ y_1^*(v_1) = & 0 & 0 & 1 & 1 & 1 \end{array}$ $\quad \left|\dfrac{[v_1, v_2]}{y_2^*[v_1, v_2]}\right| =$

| | | | | |
|---|---|---|---|---|
| [1,1] | [1,2] | [1,3] | [1,4] | [1,5] |
| 0 | 0 | 1 | 1 | 1 |
| [2,1] | [2,2] | [2,3] | [2,4] | [2,5] |
| 1 | 1 | 1 | 1 | 1 |
| [3,1] | [3,2] | [3,3] | [3,4] | [3,5] |
| 0 | 0 | 0 | 0 | 0 |
| [4,1] | [4,2] | [4,3] | [4,4] | [4,5] |
| 0 | 0 | 0 | 0 | 0 |
| [5,1] | [5,2] | [5,3] | [5,4] | [5,5] |
| 0 | 0 | 0 | 0 | 1 |

Cycle Structures:

$\overline{y^*}=$ ..... ..... ___ ___ ___ $\quad y^*=$ ..... ..... ___ ___ ___ / ___ ___ ___ ___ ___ / ___ ___ ___ ___ ___ / ___ ___ ___ ___ ___ / ___

$([111])([112])([113])([114])([115])([121])([122])([123])([124])([125])([131]\ [132]\ \overset{II}{[133]}\ [134]\ [135])([141]\ [142]\ \overset{II}{[143]}\ [144]\ [145])([151]\ [152]\ [153]\ [154]\ [155])$

$([211]\ [212]\ \overset{II}{[213]}\ [214]\ [215])([221]\ [222]\ \overset{II}{[223]}\ [224]\ [225])([231]\ [232]\ \overset{II}{[233]}\ [234]\ [235])([241]\ [242]\ \overset{II}{[243]}\ [244]\ [245])([251]\ [252]\ [253]\ [254]\ [255])$

$([311]\ [321]\ \overset{I}{[331]}\ [341]\ [351])([312]\ [322]\ \overset{I}{[332]}\ [342]\ [352])([313]\ [323]\ \overset{I}{[333]}\ [343]\ [353])([314]\ [324]\ \overset{I}{[334]}\ [344]\ [354])([315]\ [325]\ [335]\ [345]\ [355])$

$([411]\ [421]\ \overset{I}{[431]}\ [441]\ [451])([412]\ [422]\ \overset{I}{[432]}\ [442]\ [452])([413]\ [423]\ \overset{I}{[433]}\ [443]\ [453])([414]\ [424]\ \overset{I}{[434]}\ [444]\ [454])([415]\ [425]\ [435]\ [445]\ [455])$

$([511]\ [521]\ [531]\ [541]\ [551]\ [512]\ [522]\ [532]\ [542]\ [552]\ [513]\ [523]\ \overset{II}{[533]}\ [543]\ [553]\ [514]\ [524]\ [534]\ [544]\ [554]\ [515]\ [525]\ [535]\ [545]\ [555])$

Example 4. $p=5$, $n=3$, $y=\{y_o, y_1, y_2\}$

$y_o=0$, $\begin{array}{r|c|c|c|c|c|} v_1 = & 1 & 2 & 3 & 4 & 5 \\ y_1(v_1) = & 0 & 0 & 1 & 2 & 3 \end{array}$ $\quad \left|\dfrac{[v_1, v_2]}{y_2[v_1, v_2]}\right| =$

| | | | | |
|---|---|---|---|---|
| [1,1] | [1,2] | [1,3] | [1,4] | [1,5] |
| 0 | 0 | 3 | 2 | 1 |
| [2,1] | [2,2] | [2,3] | [2,4] | [2,5] |
| 2 | 2 | 4 | 3 | 1 |
| [3,1] | [3,2] | [3,3] | [3,4] | [3,5] |
| 2 | 1 | 3 | 4 | 5 |
| [4,1] | [4,2] | [4,3] | [4,4] | [4,5] |
| 1 | 2 | 3 | 4 | 2 |
| [5,1] | [5,2] | [5,3] | [5,4] | [5,5] |
| 2 | 3 | 4 | 1 | 5 |

Cycle Structures:

$\overline{y}=$ ..... ..... ..... ___ ___ $\quad y=$ ..... ..... ___ ___ ___ / ___ ___ ___ ___ ___ / ___ ___ ___ ___ ___ / ___ ___ ___ ___ ___ / ___ ___

([111])([112])([113])([114])([115])([121])([122])([123])([124])([125])([131] [134] [132] [135] [133])([141] [143] [145] [142] [144])([151] [152] [153] [154] [155])

([211] [213] [215] [212] [214])([221] [223] [225] [222] [224])([231] [235] [234] [233] [232])([241] [244] [242] [245] [243])([251] [252] [253] [254] [255])

([311] [323] [334] [342] [351])([312] [324] [335] [343] [352])([313] [325] [331] [344] [353])([314] [321] [332] [345] [314])([315] [322] [333] [341] [355])

([511] [543] [524] [552] [532])([512] [544] [525] [553] [533])([513] [545] [521] [554] [534])([514] [541] [522] [555] [535])([515] [542] [523] [551] [531])

([411] [432] [455] [422] [444] [413] [434] [452] [424] [441] [415] [431] [454] [421] [443] [412] [433] [451] [423] [445] [414] [435] [453] [425] [442])

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